6.5 Generalized Permutations and Combinations

6.5 pg 432 # 1

In how many different ways can five elements be selected in order from a set with three elements when repetition is allowed?

We have to select five elements ($r = 5$) from a set of three elements ($n = 3$) where order matters (permutation) and repetition is allowed. Therefore, there are $n^r = 3^5 = 243$ ways.

6.5 pg 432 # 7

How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?

We have to select three elements ($r = 3$) from a set of five elements ($n = 5$) where order does not matter (combination) and repetition is allowed.
Therefore, there are $C(n + r - 1, r) = C(5 + 3 - 1, 3) = C(7, 3) = 7!/(3! \times 4!) = 35$ ways.

6.5 pg 432 # 9

A bagel shop has onion bagels, poppy seed bagels, egg bagels, salty bagels, pumpernickel bagels, sesame seed bagels, raisin bagels, and plain bagels. How many ways are there to choose

b) a dozen bagels?

   There are 8 kinds of bagels ($n = 8$), and we have to choose 12 bagels ($r = 12$). In this problem, order doesn't matter (combination) and repeats are okay (we assume that the shop won't run out of any kind of bagels).
   Therefore, there are $C(n + r - 1, r) = C(8 + 12 - 1, 12) = C(19, 12) = 19!/(12! \times 7!) = 50,388$ ways to choose.

d) a dozen bagels with at least one of each kind?

   There are 8 kinds of bagels ($n = 8$), and we have to choose 4 bagels ($r = 4$) (12 minus the 8 that are already selected to satify 'at least one of each kind' condition). In this problem, order doesn't matter (combination) and repeats are okay (we assume that the shop won't run out of any kind of bagels).
   Therefore, there are $C(n + r - 1, r) = C(8 + 4 - 1, 4) = C(11, 4) = 11!/(4! \times 7!) = 330$ ways to choose.

e) a dozen bagels with at least three egg bagels and no more than two salty bagels?

   There are 8 kinds of bagels ($n = 8$), and we have to choose 9 bagels ($r = 4$) ($12 - 3$ pre-selected egg bagles). In this problem, order doesn't matter (combination) and repeats are okay (we assume that the shop won't run out of any kind of bagels).

   We have allocated three choices to the egg bagel, but we have to figure out how to choose the salty bagels. We will add the number of ways to choose when we take exactly zero, one, and two salty bagels.

No salty bagels:
$n = 7$ (7 kinds of bagels excluding the salty bagel), $r = 9$ ($12 - 3$ egg bagels). Therefore, there are $C(n+r-1, r) = C(7+9-1, 9) = C(15, 9) = 15!/(9! \times 6!) = 455$ ways to choose.

One salty bagel:
$n = 7$ (7 kinds of bagels excluding the salty bagel), $r = 8$ ($12-3$ egg bagels -1 salty bagel). Therefore, there are $C(n+r-1, r) = C(7+8-1, 8) = C(14, 8) = 14!/(8! \times 6!) = 3,003$ ways to choose.

Two salty bagels:
$n = 7$ (7 kinds of bagels excluding the salty bagel), $r = 7$ ($12-3$ egg bagels -2 salty bagels). Therefore, there are $C(n+r-1, r) = C(7+7-1, 7) = C(13, 7) = 13!/(7! \times 6!) = 1,716$ ways to choose.

So, the answer to this problem is:
$C(15, 9) + C(14, 8) + C(13, 7) = 455 + 3,003 + 1,716 = 5,174$ ways.

6.5 pg 432 # 15

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

a) $x_1 \geq 1$?

Treat each variable as a type of bagel (from the above example) and the sum as the number of bagels we have to choose. We have a restriction (namely that $x_1 \geq 1$), so we allocate one choice to x_1, and we subtract the restriction from our total choices, i.e. $21 - 1 = 20$ choices left to make.

So we have 5 variables (or 5 types of bagels) ($n = 5$), 20 choices ($r = 21 - 1 = 20$). Order does not matter and repeats are allowed.

Therefore, there are $C(n + r - 1, r) = C(5 + 20 - 1, 20) = C(24, 20) = 24!/(20! \times 4!) = 10,626$ solutions satisfying the given condition.

b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?

Here we need each variable to be at least 2 and there are 5 variables, so we have to make $21 - 2 \times 5 = 11$ selections ($r = 11$) from five variables ($n = 5$) with repetition allowled.

Since order does not matter, there are $C(n + r - 1, r) = C(5 + 11 - 1, 11) = C(15, 11) = 15!/(11! \times 4!) = 1,365$ solutions satisfying the given condition.

6.5 pg 433 # 31

How many different strings can be made from the letters in *ABRACADABRA*, using all the letters?

We have five As, two Bs, two Rs, one C, and one D for a total of 11 letters.
We can place the 5 As in $C(11, 5)$ ways, leaving 6 positions free.
Then, we can place the 2 Bs in $C(6, 2)$ ways, leaving 4 positions free.
Then, we can place the 2 Rs in $C(4, 2)$ ways, leaving 2 positions free.
Then, we can place the 1 C in $C(2, 1)$ ways, leaving 1 position free.
And we can place the 1 D in $C(1, 1)$ way.

So we have a total of $C(11, 5) \cdot C(6, 2) \cdot C(4, 2) \cdot C(2, 1) \cdot C(1, 1)$ different ways to arrange the letters, which can be computed as:
$\frac{11!}{5!\times6!} \cdot \frac{6!}{4!\times2!} \cdot \frac{4!}{2!\times2!} \cdot \frac{2!}{1!\times1!} \cdot \frac{1!}{1!\times0!} = \frac{11!}{5!\times2!\times2!\times1!\times1!} = 83,160.$