6.3 Permutations and Combinations

6.3 pg 413 # 1

List all the permutations of $\{a, b, c\}$.

This is a permutation and repeats are not allowed. Therefore, there are $P(3,3) = \frac{3!}{0!} = 6$ permutations, which are: a, b, c ; a, c, b ; b, a, c ; b, c, a ; c, a, b ; c, b, a.

6.3 pg 413 # 3

How many permutations $\{a, b, c, d, e, f, g\}$ end with a?

This is a permutation with repeats not allowed. Additionally, the last position must be an 'a', so we have only 6 items to place. Therefore, there are $P(6,6) = \frac{6!}{0!} = 720$ permutations.

6.3 pg 413 # 7

Find the number of 5-permutations of a set with nine elements.

$P(n, r)$ where $n = 9$ and $r = 5$. $P(9, 5) = 9!/(9 - 5)! = 9!/4! = 15,120$.

6.3 pg 413 # 11

How many bit strings of length 10 contain

a) exactly four 1s?

   This is just asking us to choose 4 out of 10 slots to place 1's in. $C(10, 4) = 10!/(4! \times 6!) = (10 \times 9 \times 8 \times 7)/4! = 210$.

b) at most four 1s?

   We add up the number of bit strings of length 10 that contain zero 1s, one 1, two 1s, three 1s, and four 1s.

   $C(10, 0) + C(10, 1) + C(10, 2) + C(10, 3) + C(10, 4)$

$$\begin{aligned} &= 10!/(0! \times 10!) + 10!/(1! \times 9!) + 10!/(2! \times 8!) + 10!/(3! \times 7!) + 10!/(4! \times 6!) \\ &= 1 + 10 + 45 + 120 + 210 \\ &= 386. \end{aligned}$$

c) at least four 1s?

   We subtract from the total number of bit strings of length 10 those that have only 0, 1, 2 or 3 1s.
   $2^{10} - [C(10, 0) + C(10, 1) + C(10, 2) + C(10, 3)]$
   $= 1024 - 1 - 10 - 45 - 120 = 848$.

d) an equal number of 0s and 1s?

Choose 5 out of 10 slots to place 1s (the remaining 5 slots are filled with 0s):
$C(10, 5) = 10!/(5! \times 5!) = 252.$

6.3 pg 414 # 31

The English alphabet has 21 consonants and 5 vowels. How many strings of six lowercase letters of the English alphabet contain

a) exactly one vowel?

Number of ways to choose one vowel: $C(5, 1) = 5$ ways.
There are 6 possible positions to place the chosen vowel.
Number of ways to place consonants in the 5 other positions: 21^5 ways.
Therefore, $5 \times 6 \times 21^5 = 122, 523, 030$ strings have exactly one vowel.

b) exactly two vowels?

There are $C(6, 2) \times 5^2$ ways to place two vowels in two positions. And, we can place consonants in the four remaining positions in 21^4 ways. Therefore, $C(6, 2) \times 5^2 \times 21^4 = 15 \times 25 \times 21^4 = 72, 930, 375$ strings have exactly two vowels.

c) at least one vowel?

From all strings of six letters, exclude those that have no vowels: $26^6 - 21^6 = 223, 149, 655$ strings.

d) at least two vowels?

Subtract the answer for a) from the answer for c): $223, 149, 655-122, 523, 030 = 100, 626, 625$ strings.