5.2 Strong Induction and Well-Ordering

Strong Induction

To prove that \( P(n) \) is true for all positive integers \( n \), where \( P(n) \) is a propositional function, complete two steps:

- **Basis Step**: Verify that the proposition \( P(1) \) is true.
- **Inductive Step**: Show the conditional statement \( [P(1) \land P(2) \land \cdots \land P(k)] \rightarrow P(k+1) \) is true for all positive integers \( k \).

Generalizing Strong Induction

- Handle cases where the inductive step is valid only for integers greater than a particular integer
  - \( P(n) \) is true for \( \forall n \geq b \) (\( b \): fixed integer)
- **Basis Step**: Verify that \( P(b), P(b+1), \ldots, P(b+j) \) are true (\( j \): a fixed positive integer)
- **Inductive Step**: Show that the conditional statement \( [P(b) \land P(b+1) \land \cdots \land P(k)] \rightarrow P(k+1) \) is true for all positive integers \( k \geq b+j \)

5.2 pg 341 # 3

Let \( P(n) \) be the statement that a postage of \( n \) cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that \( P(n) \) is true for \( n \geq 8 \).

a) Show that the statements \( P(8), P(9), \) and \( P(10) \) are true, completing the basis step of the proof.

\[
\begin{align*}
8 &= 3 \cdot 1 + 5 \cdot 1 \\
9 &= 3 \cdot 3 + 5 \cdot 0 \\
10 &= 3 \cdot 0 + 5 \cdot 2
\end{align*}
\]

b) What is the inductive hypothesis of the proof?

Any value \( j \) (\( 8 \leq j \leq k \)) where \( k \geq 10 \), can be expressed as \( j = 3a + 5b \) with \( a \) and \( b \) being non-negative integers.

c) What do you need to prove in the inductive step?

Assuming the inductive hypothesis, we want to show that we can express \( k + 1 \) as \( 3a + 5b \) with \( a \) and \( b \) being nonnegative integers.

d) Complete the inductive step for \( k \geq 10 \).

Since we want to show \( P(k+1) \), we can use \( P(k-2) \), which is true by inductive hypothesis since \( 8 \leq k-2 \leq k \).
\[ k - 2 = 3a + 5b \]
\[ k - 2 + 3 = 3a + 4b + 3 \]
\[ k + 1 = 3(a + 1) + 5b \]

**Explanation:**
Our base cases: 8, 9, and 10 can generate any integer value when a multiple of three is added.

*e.g.*
\[ 8 + 3 = 11 \]
\[ 9 + 3 = 12 \]
\[ 10 + 3 = 13 \]

\[ 8 + 6 = 14 \]
\[ 9 + 6 = 15 \]
\[ 10 + 6 = 16 \]

\[ \vdots \]
Therefore, by assuming \( k - 2 \) and adding a 3-cent stamp, we can get to \( k + 1 \) cents of postage.

e) Explain why these steps show that this statement is true whenever \( n \geq 8 \).

We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer \( n \) greater than or equal to 8.

**5.2 pg 342 #7**

What amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

2 dollars can also be formed, which can be proved separately.

\[ 4 = 2 \cdot 2 + 5 \cdot 0 \]
\[ 5 = 2 \cdot 0 + 5 \cdot 1 \]
\[ 6 = 2 \cdot 3 + 5 \cdot 0 \]
\[ 7 = 2 \cdot 1 + 5 \cdot 1 \]
\[ 8 = 2 \cdot 4 + 5 \cdot 0 \]
\[ 9 = 2 \cdot 2 + 5 \cdot 1 \]
\[ 10 = 2 \cdot 5 + 5 \cdot 0 \]

Inductive hypothesis: \( P(j) = \) any value \( j \) (\( 4 \leq j \leq k \)), can be expressed as \( j = 2a + 5b \) with \( a \) and \( b \) being non-negative integers.

Basis Step: \( P(4) \) and \( P(5) \) are true (see above).

Inductive step:
Assume that for \( 5 \leq k \), \( P(k - 1) \) is true.
\[ k - 1 = 2a + 5b \]
\[ k - 1 + 2 = 2a + 5b + 2 \]
\[ k + 1 = 2(a + 1) + 5b \]
This completes the inductive step.

Therefore, by the principle of strong induction, \( P(n) \) is true for all \( n \geq 4 \).

Explanation:
From \( P(4) \) and \( P(5) \), we can add a multiple of two (using 2-dollar bills) and reach any positive integer value \( \geq 4 \).

**5.2 pg 343 # 25**

Suppose that \( P(n) \) is a propositional function. Determine for which positive integers \( n \) the statement \( P(n) \) must be true, and justify your answer, if

a) \( P(1) \) is true; for all positive integers \( n \), if \( P(n) \) is true, then \( P(n + 2) \) is true.

\[ P(1) \] is true, so \( P(1 + 2) \) is true, according to the statement.
\[ P(3) \] is true, so \( P(3 + 2) \) is true.
\[ P(5) \] is true, so \( P(5 + 2) \) is true.
\[ P(n) \] is true when \( n = 1, 3, 5, 7, 9, \ldots \)

b) \( P(1) \) and \( P(2) \) are true; for all positive integers \( n \), if \( P(n) \) and \( P(n + 1) \) are true, then \( P(n + 2) \) is true.

\[ P(1) \] and \( P(1 + 1) \) are true, so \( P(1 + 2) \) is true too.
\[ P(2) \] and \( P(2 + 1) \) are true, so \( P(2 + 2) \) is true too.
\[ P(3) \] and \( P(3 + 1) \) are true, so \( P(3 + 2) \) is true too.
\[ P(4) \] and \( P(4 + 1) \) are true, so \( P(4 + 2) \) is true too.
\[ P(n) \] is true when \( n \) is any positive integer.

c) \( P(1) \) is true; for all positive integers \( n \), if \( P(n) \) is true, then \( P(2n) \) is true.

\[ P(1) \] is true, so \( P(2 \cdot 1) \) is true.
\[ P(2) \] is true, so \( P(2 \cdot 2) \) is true.
\[ P(4) \] is true, so \( P(2 \cdot 4) \) is true.
\[ P(8) \] is true, so \( P(2 \cdot 8) \) is true.
\[ P(n) \] is true when \( n \) is an integer and a power of 2. (i.e. \( n = 2, 4, 8, 16, \ldots \))