12.1 Boolean Functions

Boolean algebra provides the operations and rules for working with the set $\{0, 1\}$.

Boolean Complement

- $\overline{x} \equiv \neg x$
- $\overline{0} = 1$
- $\overline{1} = 0$

Boolean Sum

- $x + y \equiv x \vee y$
- $0 + 0 = 0$
- $0 + 1 = 1$
- $1 + 0 = 1$
- $1 + 1 = 1$

Boolean Product

- $x \cdot y \equiv x \wedge y$
- $0 \cdot 0 = 0$
- $0 \cdot 1 = 0$
- $1 \cdot 0 = 0$
- $1 \cdot 1 = 1$

Boolean Functions

Let $B = \{0, 1\}$. Then $B^n = \{(x_1, x_2, \ldots, x_n) | x_i \in B$ for $1 \leq i \leq n\}$ is the set of all possible n-tuples of 0s and 1s. The variable x is called a *Boolean variable* if it assumes values only from B, that is, if its only possible values are 0 and 1. A function from B^n to B is called a *Boolean function of degree n*.

Boolean Expressions

The *Boolean expressions* in the variables, $x_1, x_2, \ldots, x_n$ are defined recursively as $0, 1, x_1, x_2, \ldots, x_n$ are Boolean expressions; if E_1 and E_2 are Boolean expressions, then $\overline{E}_1$, (E_1E_2), and $(E_1 + E_2)$ are Boolean expressions. Each Boolean expression represents a Boolean function.

Boolean Identities

Identity	Name
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \bar{x}\bar{y}$	De Morgan's laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

Duality

The *dual* of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

Abstract Definition of Boolean Algebra

A general *Boolean algebra* is a set B with elements 0 and 1, two binary operators $\wedge$ and $\vee$, and a unary operator $\neg$ that satisfies the following laws for all x, y, and z in B:

- Identity laws:
  - $x \vee 0 = x$
  - $x \wedge 1 = x$
- Complement laws:
  - $x \vee \neg x = 1$
  - $x \wedge \neg x = 0$
- Associative laws:
  - $(x \vee y) \vee z = x \vee (y \vee z)$
  - $(x \wedge y) \wedge z = x \wedge (y \wedge z)$
- Commutative laws:

  - $x \vee y = y \vee x$
  - $x \wedge y = y \wedge x$

- Distributive laws:
  - $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
  - $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

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Use a table to express the values of each of these Boolean functions.

a) $F(x, y, z) = \overline{x}y$

x	y	z	$\overline{x}$	$\overline{x}y$
1	1	1	0	0
1	1	0	0	0
1	0	1	0	0
1	0	0	0	0
0	1	1	1	1
0	1	0	1	1
0	0	1	1	0
0	0	0	1	0

b) $F(x, y, z) = x + yz$

x	y	z	yz	$x + yz$
1	1	1	1	1
1	1	0	0	1
1	0	1	0	1
1	0	0	0	1
0	1	1	1	1
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

c) $F(x, y, z) = x\overline{y} + \overline{(xyz)}$

x	y	z	$\overline{y}$	$x\overline{y}$	xyz	$\overline{(xyz)}$	$x\overline{y} + \overline{(xyz)}$
1	1	1	0	0	1	0	0
1	1	0	0	0	0	1	1
1	0	1	1	1	0	1	1
1	0	0	1	1	0	1	1
0	1	1	0	0	0	1	1
0	1	0	0	0	0	1	1
0	0	1	1	0	0	1	1
0	0	0	1	0	0	1	1

d) $F(x, y, z) = x(yz + \bar{y}\bar{z})$

x	y	z	$\overline{y}$	$\overline{z}$	yz	$\bar{y}\bar{z}$	$yz + \bar{y}\bar{z}$	$x(yz + \bar{y}\bar{z})$
1	1	1	0	0	1	0	1	1
1	1	0	0	1	0	0	0	0
1	0	1	1	0	0	0	0	0
1	0	0	1	1	0	1	1	1
0	1	1	0	0	1	0	1	0
0	1	0	0	1	0	0	0	0
0	0	1	1	0	0	0	0	0
0	0	0	1	1	0	1	1	0

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What values of the Boolean variables x and y satisfy $xy = x + y$?

$xy = x + y$ is only satisfied if and only if $x = y$. We can easily see that when $x = y = 0$ or when $x = y = 1$ is when the equation is satisfied.

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Prove the absorption law $x + xy = x$ using the other laws.

$x + xy$
$= x \cdot 1 + xy$ by identity law
$= x(1 + y)$ by distributive law
$= x(y + 1)$ by commutative law
$= x \cdot 1$ by domination law
$= x$ by identity law

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Show that $x\overline{y} + y\overline{z} + \overline{x}z = \overline{x}y + \overline{y}z + x\overline{z}$.

Simply create a table and list the values.

x	y	z	$\overline{x}$	$\overline{y}$	$\overline{z}$	$x\overline{y}$	$y\overline{z}$	$\overline{x}z$	$\overline{x}y$	$\overline{y}z$	$x\overline{z}$	$x\overline{y} + y\overline{z} + \overline{x}z$	$\overline{x}y + \overline{y}z + x\overline{z}$
1	1	1	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	1	0	1	0	0	0	1	1	1
1	0	1	0	1	0	1	0	0	0	1	0	1	1
1	0	0	0	1	1	1	0	0	0	0	1	1	1
0	1	1	1	0	0	0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	1	0	1	0	0	1	1
0	0	1	1	1	0	0	0	1	0	1	0	1	1
0	0	0	1	1	1	0	0	0	0	0	0	0	0