## 8.6 Applications of Inclusion-Exclusion

**Alternative form of inclusion-exclusion**

The alternative form of inclusion-exclusion is used to find the number of elements in a set that have none of $n$ properties $P_1, P_2, ..., P_n$. Let $N(P_1'P_2'...P_n')$ denote the number of elements that have none of the properties $P_1, P_2, ..., P_n$. Then we'll have:

$$N(P_1'P_2'...P_n') = N - |A_1 \cup A_2 \cup ... \cup A_n|$$

By the inclusion-exclusion principle, we can see that

$$N(P_1'P_2'...P_n') = N - \sum_{1 \le i \le n} |A_i| + \sum_{1 \le i < j \le n} |A_i \cap A_j| - \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + ... + (-1)^n |A_1 \cap A_2 \cap ... \cap A_n|$$

**Number of Onto Functions**

**Reviewing onto function:** A function $f$ from $A$ to $B$ is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. (Section 2.3 pg. 143)

**Theorem 1:** Let $m$ and $n$ be positive integers with $m \ge n$. Then, there are

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - ... + (-1)^{n-1}C(n, n-1) \cdot 1^m$$

onto functions from a set with $m$ elements to a set with $n$ elements.

**8.6 pg. 564 # 1**

Suppose that in a bushel of 100 apples there are 20 that have worms in them and 15 that have bruises. Only those apples with neither worms nor bruises can be sold. If there are 10 bruised apples that have worms in them, how many of the 100 apples can be sold?

We are looking for the number of apples that have neither of the properties of having worms or having bruises. We can simply apply the the alternative form of inclusion-exclusion here.

Let $P_1$ denote the property of having worms in them and $P_2$ denote the property of having bruises. We know that $N(P_1) = 20, N(P_2) = 15$, and $N(P_1P_2) = 10$. By using the alternative form of inclusion-exclusion, our equation is $N(P_1'P_2') = N - N(P_1) - N(P_2) + N(P_1P_2) = 100 - 20 - 15 + 10 = 75$. We can sell 75 apples.

**8.6 pg. 564 # 3**

How many solutions does the equation $x_1 + x_2 + x_3 = 13$ have where $x_1, x_2$, and $x_3$ are nonnegative integers less than 6?

Let $P_1, P_2, P_3$ be the property of the solution when
$P_1 = x_1 \ge 6$
$P_2 = x_2 \ge 6$

$P_3 = x_3 \geq 6$

Then the number of total solutions $N = C(3 + 13 - 1, 13) = C(15, 13) = 105$
We need to find the remaining number of solutions:
$N(P_1) = C(3 + (13 - 6) - 1, (13 - 6)) = C(9, 7) = 36$
$N(P_2) = C(3 + (13 - 6) - 1, (13 - 6)) = C(9, 7) = 36$
$N(P_3) = C(3 + (13 - 6) - 1, (13 - 6)) = C(9, 7) = 36$
$N(P_1P_2) = C(3 + (13 - 6 - 6) - 1, (13 - 6 - 6)) = C(3, 1) = 3$
$N(P_2P_3) = C(3 + (13 - 6 - 6) - 1, (13 - 6 - 6)) = C(3, 1) = 3$
$N(P_1P_3) = C(3 + (13 - 6 - 6) - 1, (13 - 6 - 6)) = C(3, 1) = 3$
$N(P_1P_2P_3) = 0$

We need to find $N(P_1'P_2'P_3')$.
$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_2P_3) + N(P_1P_3) - N(P_1P_2P_3)$
$= 105 - 36 - 36 - 36 + 3 + 3 + 3 - 0$
$= 6$

There are 6 solutions to the problem.

**8.6 pg. 564 # 9**

How many ways are there to distribute six different toys to three different children such that each child gets at least one toy?

This problem is an onto function where the set of toys is assigned to the set of children, so we can apply Theorem 1 where $m = 6$ and $n = 3$.
$3^6 - C(3, 1)2^6 + C(3, 2)1^6$
$= 729 - 3(64) + 3(1)$
$= 729 - 192 + 3$
$= 540$

There are 540 ways to distribute six toys to the three children if each child gets at least one toy.