10.2 Graph Terminology and Special Types of Graphs

Undirected Graph

Adjacent/Neighbors and Incident Edge

Two vertices u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G if u and v are endpoints of an edge e of G. Such an edge e is called *incident* with the vertices u and v and e is said to *connect* u and v.

Neighborhood

The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V , we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.

Degree of a Vertex

The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

- A vertex of degree zero is called *isolated*.
- A vertex is a *pendant* if and only if it has a degree one.

Handshaking Theorem

Let $G = (V, E)$ be an undirected graph with m edges. Then $2m = \sum_{v \in V} \deg(v)$.

Directed Graph

Adjacency

When (u, v) is an edge of the graph G with directed edges, u is said to be *adjacent to* v and v is said to be *adjacent from* u. The vertex u is called the *initial vertex* of (u, v), and v is called the *terminal* or *end vertex* of (u, v). The initial vertex and terminal vertex of a loop are the same.

Degree of a Vertex

In a graph with directed edges the *in-degree* of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The *out-degree* of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

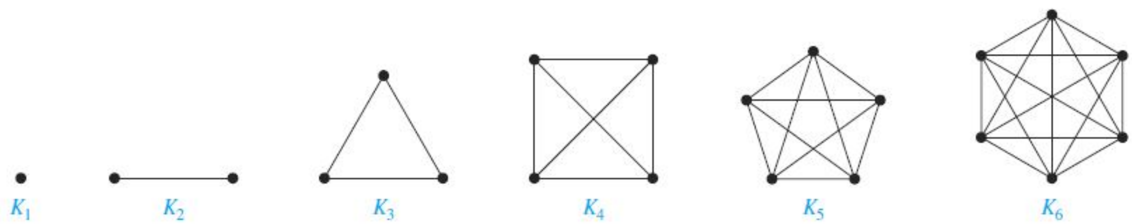
Handshaking Theorem for Directed Graphs (Theorem 3)

Let $G = (V, E)$ be a graph with directed edges. Then $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$.

Special Graphs

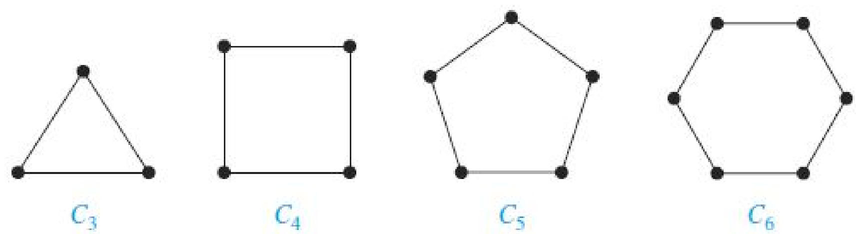
Complete Graphs

A complete graph on n vertices, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. Has $\frac{n(n-1)}{2}$ edges.



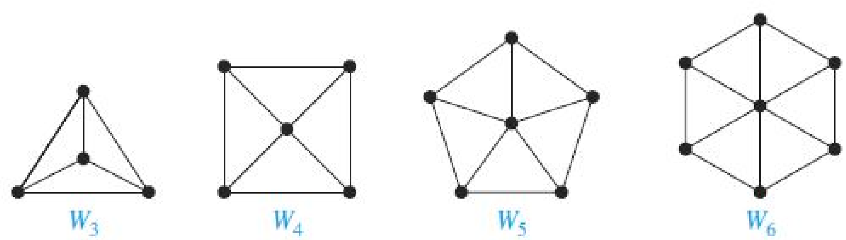
Cycles

A *cycle* $C_n, n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. Has n edges.



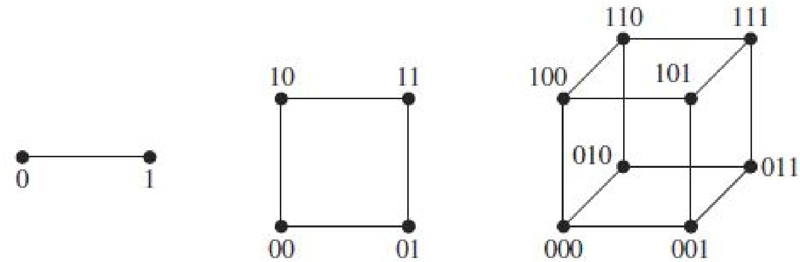
Wheels

We obtain a *wheel* W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges. Has $2n$ edges and $n + 1$ vertices.



n-Cubes

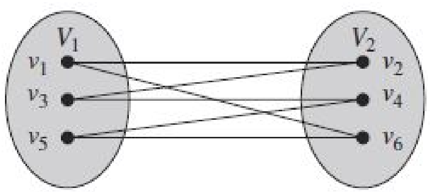
An *n-dimensional hypercube*, or *n-cube*, denoted by Q_n, is a graph that has vertices representing the 2^n bit strings of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. Has 2^n vertices and $n2^{n-1}$ edges (note that there are 0 edges in Q_0).



Q_1 Q_2 Q_3

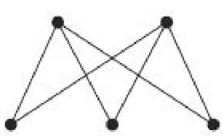
Bipartite Graphs

A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.

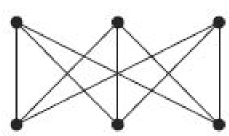


Complete Bipartite Graphs

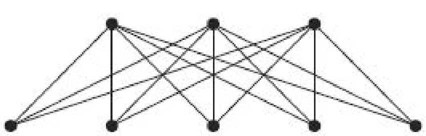
A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between every pair of vertices if and only if one vertex in the pair is in the first subset and the other vertex is in the second subset.



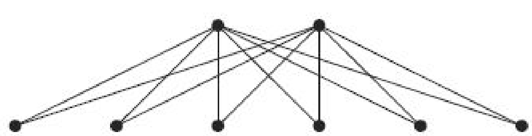
$K_{2,3}$



$K_{3,3}$



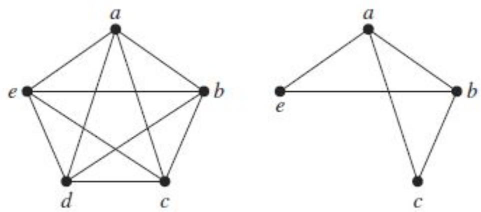
$K_{3,5}$



$K_{2,6}$

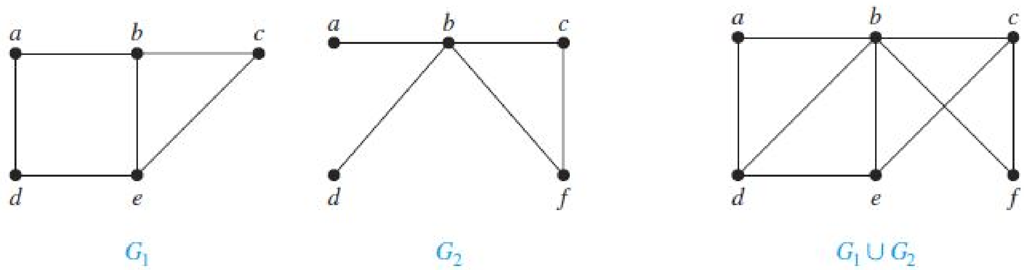
Subgraphs

A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.



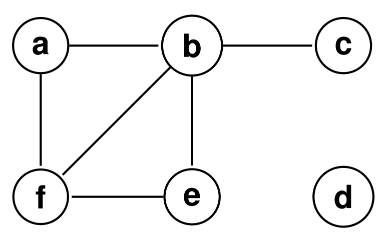
Graph Union

The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.



10.2 pg. 665 # 1

Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices.



We have 6 vertices and 6 edges. deg(a) = 2, deg(b) = 4, deg(c) = 1, deg(d) = 0, deg(e) = 2, deg(f) = 3. c is a pendant and d is isolated.

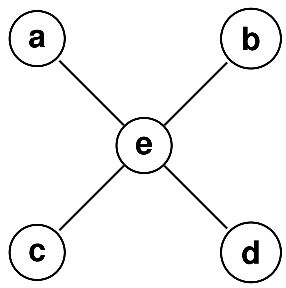
10.2 pg. 665 # 13

What does the degree of a vertex represent in an academic collaboration graph? What does the neighborhood of a vertex represent? What do isolated and pendant vertices represent?

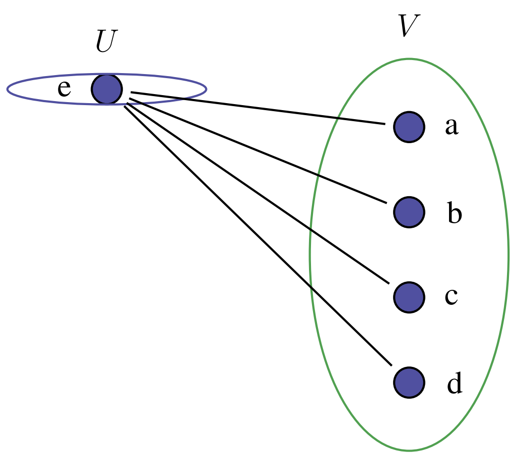
The degree of a vertex in a academic collaboration graph represents the number of collaborators a person had. A neighborhood of a vertex would represents all the co-authors that person has published with. An isolated vertex would represent a person that published a paper with no co-authors. A pendant vertex would represent the person published with one other co-author.

10.2 pg. 665 # 21

Determine whether the graph is bipartite.

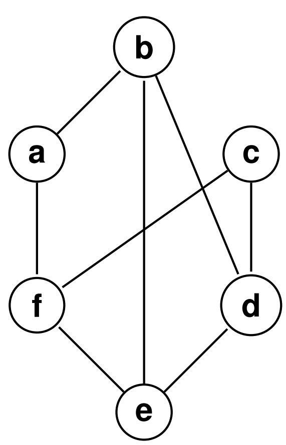


Yes, this graph is a bipartite. We will draw the bipartite graph to illustrate this.



10.2 pg. 666 # 25

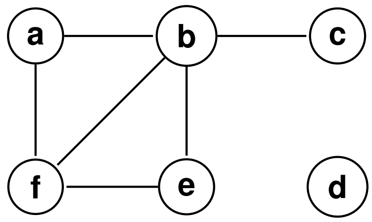
Determine whether the graph is bipartite.



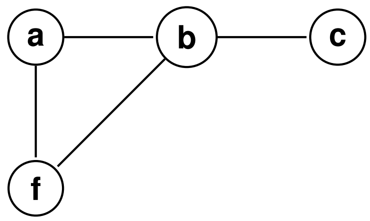
This graph is not bipartite. If we consider the triangle $\triangle bde$, we would have vertices that are joined by the other two. By pigeonhole principle, at least two of them must be in the same bipartition. If two of the vertices are in the same bipartition, then there will be an edge connecting between them. Since this violates the definition of a bipartite graph, this is graph is not bipartite.

10.2 pg. 666 # 33

For the following graph G find

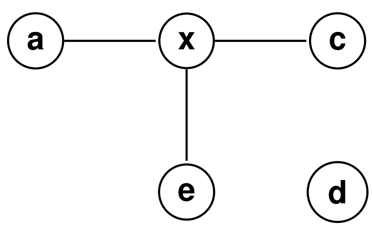


a) a subgraph induced by the vertices a, b, c, and f.



b) the new graph G_1 obtained from G by contracting the edge connecting b and f

We will contract b and f together and call it vertex x



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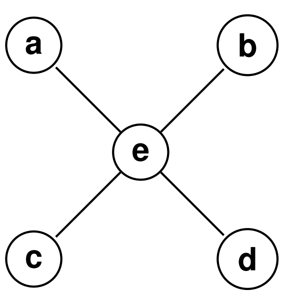
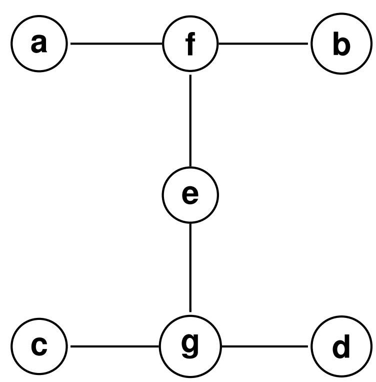
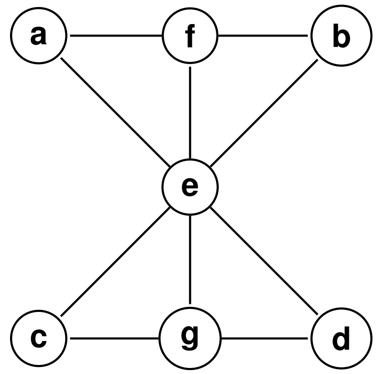
Find the union of the given pair of simple graphs.



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