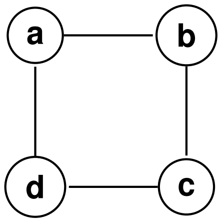
10.3 Representing Graphs and Graph Isomorphism

Adjacency Lists

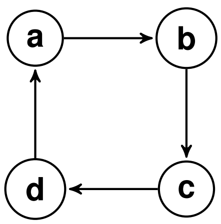
- Can be used to represent a graph with no multiple edges
- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	Adjacent Vertex
a	b, d
b	a, c
c	b, d
d	a, c

Directed Adjacency Lists

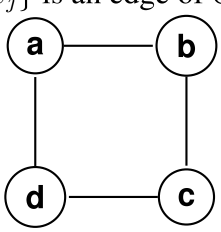
- 1 row per vertex, listing the terminal vertices of each edge incident from that vertex.



Initial Vertex	Terminal Vertices
a	b
b	c
c	d
d	a

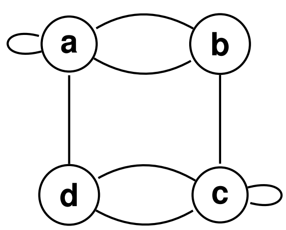
Adjacency Matrix

Let the *adjacency matrix* $A_G = [a_{ij}]$ of a graph G is the $n \times n$ ($n = |V|$) zero-one matrix, where $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G, and is 0 otherwise.



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

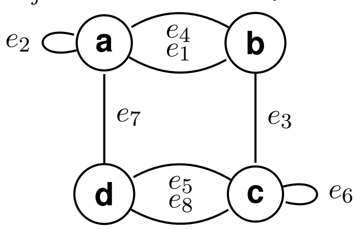
- Can extend to graphs with loops and multiple edges by letting each matrix elements be the number of links (possibly > 1) between the nodes.



$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

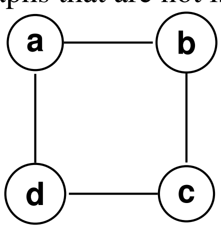
Let $G = (V, E)$ be an undirected graph with $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$ where $m_{ij} = 1$ if e_j is incident with v_i, and is 0 otherwise.

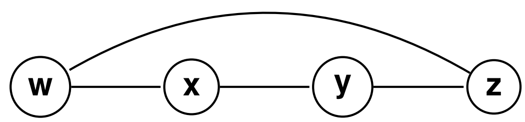


	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
a	1	1	0	1	0	0	1	0
b	1	0	1	1	0	0	0	0
c	0	0	1	0	1	1	0	1
d	0	0	0	0	1	0	1	1

Graph Isomorphism

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an *isomorphism*. Two simple graphs that are not isomorphic are called *nonisomorphic*.





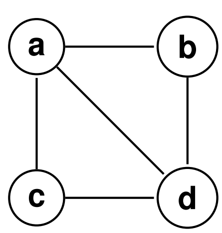
Graph Invariants

Properties preserved by isomorphism of graphs.

- must have the same number of vertices
- must have the same number of edges
- must have the same number of vertices with degree k
- for every proper subgraph g of one graph, there must be a proper subgraph of the other graph that is isomorphic of g

10.3 pg. 675 # 1 & # 5

Use an adjacency list and adjacency matrix to represent the given graph.

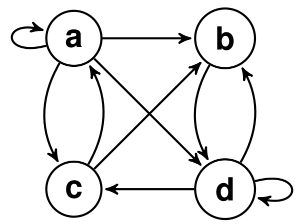


Vertex	Adjacent vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

10.3 pg. 675 # 3 & # 7

Use an adjacency list and adjacency matrix to represent the given graph.



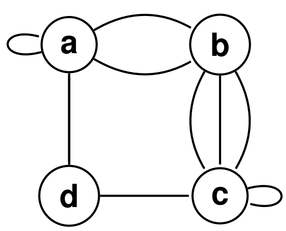
Initial Vertex	Terminal Vertex
a	a, b, c, d
b	d
c	a, b
d	b, c, d

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

10.3 pg. 675 # 17

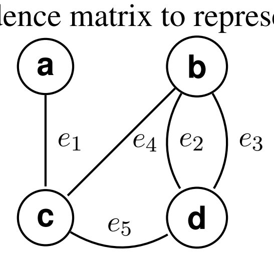
Draw an undirected graph represented by the given adjacency matrix.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$



10.3 pg. 676 # 27

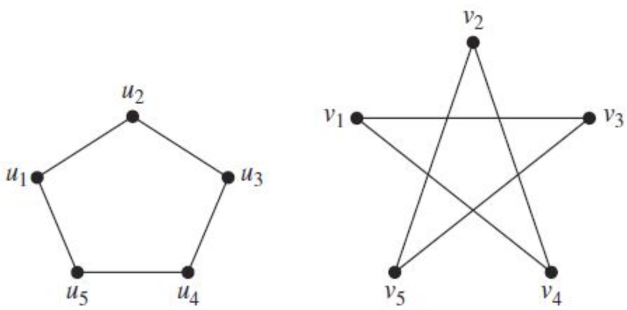
Use an incidence matrix to represent the graph.



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

10.3 pg. 667 # 35

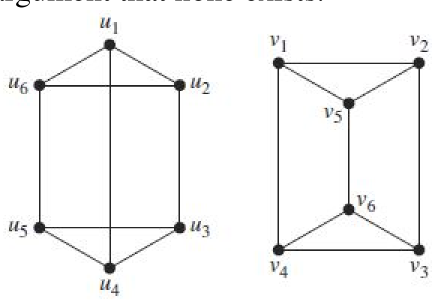
Determine whether the pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



This graph is isomorphic. One isomorphism is $f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_5, f(u_4) = v_2$, and $f(u_5) = v_4$.

10.3 pg. 667 # 39

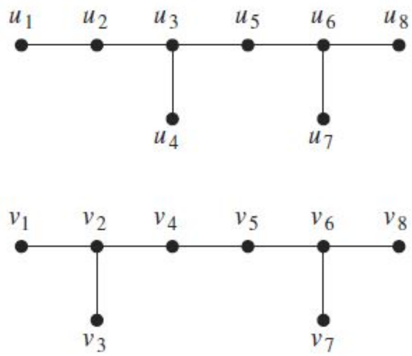
Determine whether the pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



This graph is isomorphic. One isomorphism is $f(u_1) = v_5, f(u_2) = v_2, f(u_3) = v_3, f(u_4) = v_6, f(u_5) = v_4$, and $f(u_6) = v_1$.

10.3 pg. 667 # 41

Determine whether the pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



These graphs are not isomorphic. Consider the two vertices of degree 3 (u_3 and u_6) in the first graph. They are within the neighborhood of u_5. However, in the second graph, the two vertices of degree 3 are not within the neighborhood of a common vertex. Thus, they are not isomorphic.