9.3 Representing Relations

Representing Relations using Zero-One Matrices

Let R be a relation from $A = \{a_1, a_2, \dots, a_m\}$ to $B = \{b_1, b_2, \dots, b_n\}$. The relation R can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Reflexive in a Zero-One Matrix

Let R be a binary relation on a set and let M be its zero-one matrix. R is reflexive if and only if $M_{ii} = 1$ for all i. In other words, all elements are equal to 1 on the main diagonal.

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Symmetric in a Zero-One Matrix

Let R be a binary relation on a set and let M be its zero-one matrix. R is symmetric if and only if $M = M^t$. In other words, $M_{ij} = M_{ji}$ for all i and j.

$$\begin{bmatrix} \ddots & 1 & & \\ 1 & \ddots & 0 & \\ & 0 & \ddots & 1 \\ & & 1 & \ddots \end{bmatrix}$$

Antisymmetric in a Zero-One Matrix

Let R be a binary relation on a set and let M be its zero-one matrix. R is antisymmetric if and only if $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$.

$$\begin{bmatrix} \ddots & 1 & & \\ 0 & \ddots & 0 & \\ & 0 & \ddots & 0 \\ & & 1 & \ddots \end{bmatrix}$$

Join

If M_1 is the zero-one matrix for R_1 and M_2 is the zero-one matrix for R_2 then the join of M_1 and M_2, i.e. $M_1 \vee M_2$, is the zero-one matrix for $R_1 \cup R_2$.

Meet

If M_1 is the zero-one matrix for R_1 and M_2 is the zero-one matrix for R_2 then the meet of M_1 and M_2, i.e. $M_1 \wedge M_2$, is the zero-one matrix for $R_1 \cap R_2$.

Composition of Relations

Let M_1 be the zero-one matrix for R_1 and M_2 be the zero-one matrix for R_2. Then, the Boolean product of two matrices M_1 and M_2, denoted $M_1 \odot M_2$, is the zero-one matrix for the composite of R_1 and R_2, $R_2 \circ R_1$.

$$(M_1 \odot M_2)_{ij} = \bigvee_{k=1}^{n} [(M_1)_{ik} \wedge (M_2)_{kj}]$$

Digraph

A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the *terminal vertex* of this edge.

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Represent each of these relations on $\{1, 2, 3\}$ with a matrix (with the elements of this set listed in increasing order).

a $\{(1, 1), (1, 2), (1, 3)\}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b $\{(1, 2), (2, 1), (2, 2), (3, 3)\}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

d $\{(1, 3), (3, 1)\}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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List the ordered pairs in the relations on $\{1, 2, 3\}$ corresponding to these matrices (where the rows and columns correspond to the integers listed in increasing order).

a $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$

b $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\{(1, 2), (2, 2), (3, 2)\}$

c $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2), (3, 3)\}$

9.3 pg. 596 # 9

How many nonzero entries does the matrix representing the relation R on $A = \{1, 2, 3, \ldots, 100\}$ consisting of the first 100 positive integers if R is

a $\{(a, b)|a > b\}$?

We'll start from the bottom of the matrix. We know that that 100 is greater than 99 positive integers. We also know that 99 is greater than 98 positive integers. Following this line of thought, it's easy to see that the total number of elements is:

$99 + 98 + \ldots + 1 = \sum_{i=1}^{99} i = \dfrac{99(99 + 1)}{2} = 4950.$

b $\{(a, b)|a \neq b\}$?

We first note that the total amount of elements in the matrix is $100^2 = 10000$.
We know that we have a 1 for every element in the matrix besides the main diagonal. The main diagonal is 100 elements, so the total number of nonzero entries is:
$10000 - 100 = 9900$

c $\{(a, b)|a = b + 1\}$?

This relation means that there is a 1 at every position below the main diagonal. Therefore, the answer is 99.

d $\{(a, b)|a = 1\}$?

Only the top row has a 1 in every position. This would mean we have 100 nonzero entries.

e $\{(a, b)|ab = 1\}$?

The only valid entry is $(1, 1)$, so our answer is 1.

9.3 pg. 596 # 13

Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the matrix representing

a R^{-1}

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

b $\overline{R}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

c R^2

$$\begin{bmatrix} (0 \wedge 0) \vee (1 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 1) \vee (1 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 1) \\ (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

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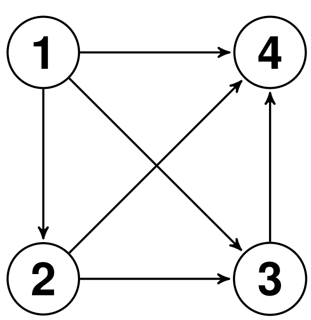
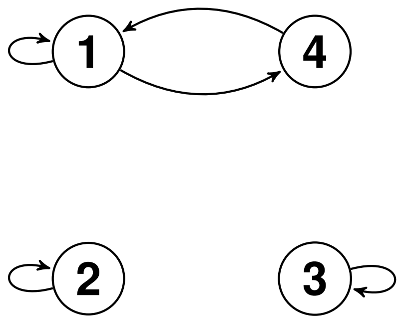
Draw the directed graphs representing each of the relations

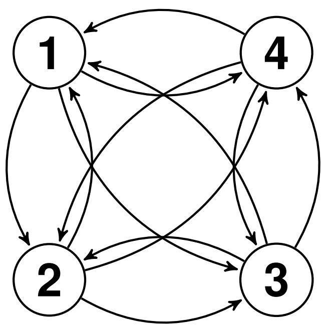
a $\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$



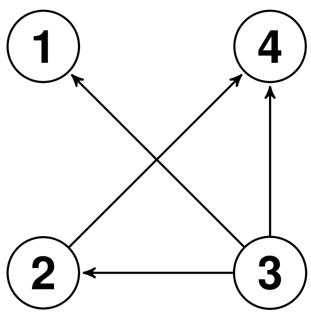
b $\{(1,1),(1,4),(2,2),(3,3),(4,1)\}$



c $\{(1,2),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,2),(3,4),(4,1),(4,2),(4,3)\}$

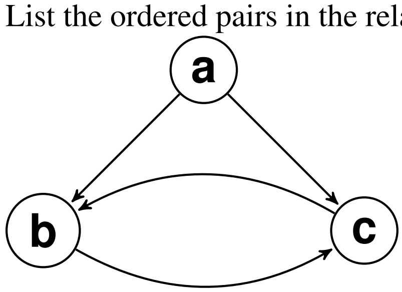


d $\{(2,4),(3,1),(3,2),(3,4)\}$



9.3 pg. 597 # 23

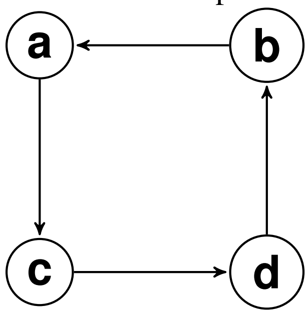
List the ordered pairs in the relations represented by the directed graph.



$\{(a,b),(a,c),(b,c)(c,b)\}$

9.3 pg. 597 # 25

List the ordered pairs in the relations represented by the directed graph.



$\{(a, c), (b, a), (c, d)(d, b)\}$

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Determine whether the relation represented by the digraph shown in Exercises 23 and 25 are reflexive, irreflexive, symmetric, antisymmetric, and/or transitive.

\# 23) Not reflexive because every vertex does not have a self loop.
Is irreflexive because there are no self loops.
Not symmetric because we do not have (b, a) for (a, b).
Not antisymmetric because we have (b, c) and (c, b).
Not transitive because the path for (b, c), (c, b) from b to b is missing the edge(b, b).

\# 25) Not reflexive because every vertex does not have a self loop.
Is irreflexive because there are no self loops.
Not symmetric because we do not have (c, a) for (a, c).
Is antisymmetric because there are no edges that go in the opposite direction for each edge.
Not transitive because we do not have (a, d) for the edges (a, c) and (c, d).