9.4 Closure of Relations

Reflexive Closure

The \textit{reflexive closure} of a relation \( R \) on \( A \) is obtained by adding \((a, a)\) to \( R \) for each \( a \in A \).

Symmetric Closure

The \textit{symmetric closure} of \( R \) is obtained by adding \((b, a)\) to \( R \) for each \((a, b) \in R\).

Transitive Closure

The \textit{transitive closure} of \( R \) is obtained by repeatedly adding \((a, c)\) to \( R \) for each \((a, b) \in R \) and \((b, c) \in R\).

Paths and Circuits in Directed Graphs

A \textit{path} from \( a \) to \( b \) in the directed graph \( G \) is a sequence of edges \((x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)\) in \( G \), where \( n \) is a nonnegative integer, and \( x_0 = a \) and \( x_n = b \), that is, a sequence of edges where the terminal vertex of an edge is the same as the initial vertex in the next edge in the path. This path is denoted by \( x_0, x_1, x_2, \ldots, x_{n-1}, x_n \) and has \textit{length} \( n \). We view the empty set of edges as a path of length zero from \( a \) to \( a \). A path of length \( n \geq 1 \) that begins and ends at the same vertex is called a \textit{circuit} or \textit{cycle}.

Path in a Relation

\textbf{Theorem 1:} Let \( R \) be a relation on a set \( A \). There is a path of length \( n \), where \( n \) is a positive integer, from \( a \) to \( b \) if and only if \((a, b) \in R^n\).

Connectivity Relation A.K.A. Transitive Closures

Let \( R \) be a relation on a set \( A \). The \textit{connectivity relation} \( R^* \) consists of the pairs \((a, b)\) such that there is a path of length at least one from \( a \) to \( b \) in \( R \).

In other words:

\[ R^* = \bigcup_{n=1}^{\infty} R^n \]

where \( R^n \) consists of the pairs \((a, b)\) such that there is a path of length \( n \) from \( a \) to \( b \).

\textbf{Theorem 2:} The transitive closure of a relation \( R \) equals the connectivity relation \( R^* \).

\textbf{Theorem 3:} Let \( M_R \) be the zero-one matrix of the relation \( R \) on a set with \( n \) elements. Then the zero-one matrix of the transitive closure \( R^* \) is

\[ M_{R^*} = M_R \lor M_R^{[2]} \lor M_R^{[3]} \lor \ldots \lor M_R^{[n]} \]
Simple Algorithm for Computing Transitive Closure

This algorithm shows how to compute the transitive closure. Runs in $O(n^4)$ bit operations.

**Algorithm** \(\text{transitive\_closure}(M_R \text{: zero-one } n \times n \text{ matrix})\)

\[
A = M_R \\
B = A \\
\text{for } i = 2 \text{ to } n \text{ do} \\
\hspace{1em} A = A \odot M_R \\
\hspace{1em} B = B \lor A \\
\text{end for} \\
\text{return } B \{ B \text{ is the zero-one matrix for } R^* \}
\]

Warshall’s Algorithm

Warshall’s algorithm is a faster way to compute transitive closure. Runs in $O(n^3)$ bit operations.

**Algorithm** \(\text{Warshall}(M_R \text{: zero-one } n \times n \text{ matrix})\)

\[
W = M_R \\
\text{for } k = 1 \text{ to } n \text{ do} \\
\hspace{1em} \text{for } i = 1 \text{ to } n \text{ do} \\
\hspace{2em} \text{for } j = 1 \text{ to } n \text{ do} \\
\hspace{3em} w_{ij} = w_{ij} \lor (w_{ik} \land w_{kj}) \\
\hspace{2em} \text{end for} \\
\hspace{1em} \text{end for} \\
\text{end for} \\
\text{return } W \{ W = [w_{ij}] \text{ is the zero-one matrix for } R^* \}
\]

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Let \(R\) be the relation on the set \(\{0, 1, 2, 3\}\) containing the ordered pairs \((0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0)\). Find the

a) reflexive closure of \(R\)

We need to add \((a, a)\) in \(R\) to make a reflexive closure.
\[
\{(0, 0), (0, 1), (1, 1), (1, 2), (2, 0), (2, 2), (3, 0), (3, 3)\}
\]

b) symmetric closure of \(R\)

We need to add \((b, a)\) for each \((a, b)\) in \(R\) to make a symmetric closure.
\[
\{(0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0)\}
9.4 pg. 607 # 5
For the directed graph shown

\[
\begin{array}{cccc}
  a & b & c & d \\
  \downarrow & & \downarrow & \\
  b & c & & d \\
  \uparrow & & \uparrow & \\
  a & & c & d \\
\end{array}
\]

a) Find the reflexive closure

\[
\begin{array}{cccc}
  a & b & c & d \\
  \downarrow & & \downarrow & \\
  b & c & & d \\
  \uparrow & & \uparrow & \\
  a & & c & d \\
\end{array}
\]

b) Find the symmetric closure

\[
\begin{array}{cccc}
  a & b & c & d \\
  \downarrow & & \uparrow & \\
  b & c & & d \\
  \uparrow & & \downarrow & \\
  a & & c & d \\
\end{array}
\]

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Use Algorithm 1 to find the transitive closure of these relations on \{1, 2, 3, 4\}.

a) \{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}

Transitive Closure
\[M_R^+ = M_R \lor M_R^{[2]} \lor M_R^{[3]} \lor M_R^{[4]}\]
b) \{ (2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3) \}

**Transitive Closure**

\[ M_{R^*} = M_R \lor M_R^2 \lor M_R^3 \lor M_R^4 = \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\lor
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\lor
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\lor
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

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Use Warshall’s algorithm to find the transitive closure of these relations on \{1, 2, 3, 4\}.

### a) \{ (1, 2), (2, 1), (2, 3), (3, 4), (4, 1) \}

\[
W_0 =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix}
W_1 =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0
\end{bmatrix}
W_2 =
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{bmatrix}
W_3 =
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
W_4 =
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix}
\]

### b) \{ (2, 1), (2, 3), (3, 1), (3, 4), (4, 1), (4, 3) \}

\[
W_0 =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
W_1 =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
W_2 =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
W_3 =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
W_4 =
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1
\end{bmatrix}
\]