9.5 Equivalence Relations

A relation on a set $A$ is called an equivalence relation if it is reflexive, symmetric, and transitive.

Equivalence Classes

Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to an element $a$ of $A$ is called the equivalence class of $a$. The equivalence class of $a$ with respect to $R$ is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript $R$ and write $[a]$ for this equivalence class. This can be represented by:

$$[a]_R = \{s| (a, s) \in R\}$$

Partitions and Equivalence Classes

Let $A_1, A_2, \ldots, A_i$ be a collection of subsets of $S$. Then the collection forms a partition of $S$ if the subsets are nonempty, disjoint and exhaust $S$:

- $A_i \neq \emptyset$ for $i \in I$
- $A_i \cap A_j = \emptyset$ if $i \neq j$
- $\bigcup_{i \in I} A_i = S$

**Theorem 1:** Let $R$ be an equivalence relation on a set $A$. These statements for elements $a$ and $b$ of $A$ are equivalent:

- $aRb$
- $[a] = [b]$
- $[a] \cap [b] \neq \emptyset$

**Theorem 2:** Let $R$ be an equivalence relation on a set $S$. Then the equivalence classes of $R$ form a partition of $S$. Conversely, given a partition $\{A_i|i \in I\}$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i, i \in I$, as its equivalence classes.

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Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

This is an equivalence relation because it is reflexive, symmetric, and transitive.

b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2)(3, 3)\}$

This is not an equivalence relation because it is neither reflexive nor transitive. Missing $(1, 1)$ for reflexive and missing $(0, 3)$ for the path $(0, 2), (2, 3)$ for transitive.
c) \{ (0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3) \}

This is an equivalence relation because it is reflexive, symmetric, and transitive.

d) \{ (0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3) \}

This is not an equivalence relation because it is not transitive. Missing (1, 2) for the path (1, 3), (3, 2).

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Suppose that \( A \) is a nonempty set, and \( f \) is a function that has \( A \) as its domain. Let \( R \) be the relation on \( A \) consisting of all ordered pairs \( (x, y) \) such that \( f(x) = f(y) \). Show that \( R \) is an equivalence relation on \( A \).

This relation is reflexive because it is obvious that \( f(x) = f(x) \) for all \( x \in A \).

This relation is symmetric because if \( f(x) = f(y) \), then it follows that \( f(y) = f(x) \).

This relation is transitive because if we have \( f(x) = f(y) \) and \( f(y) = f(z) \), then it follows that \( f(x) = f(z) \).

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Determine whether the relation with the directed graph shown is an equivalence relation.

Not an equivalence relation because we are missing the edges \((c, d)\) and \((d, c)\) for transitivity.

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Determine whether the relation with the directed graph shown is an equivalence relation.

Not an equivalence relation because we are missing the edges \((a, c)\), \((c, a)\), \((b, d)\), and \((d, b)\) for transitivity.
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What is the congruence class \([n]_5\) (that is, the equivalence class of \(n\) with respect to congruence modulo 5) when \(n\) is

a) 2?
\[ [2]_5 = \{i|\, i \equiv 2 \pmod{5}\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\} \]

b) 3?
\[ [3]_5 = \{i|\, i \equiv 3 \pmod{5}\} = \{\ldots, -7, -2, 3, 8, 13, \ldots\} \]

c) 6?
\[ [6]_5 = \{i|\, i \equiv 6 \pmod{5}\} = \{\ldots, -9, -4, 1, 6, 11, \ldots\} \]

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Which of these collections of subsets are partitions of \(\{1, 2, 3, 4, 5, 6\}\)?

a) \(\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}\)

Not a partition because these sets are not pairwise disjoint. The element 2 and 4 appear in two of these sets.

b) \(\{1\}, \{2, 3, 6\}, \{4\}, \{5\}\)

This is a partition.

c) \(\{2, 4, 6\}, \{1, 3, 5\}\)

This is a partition.

d) \(\{1, 4, 5\}, \{2, 6\}\)

Not a partition because it is missing the element 3 in any of the sets.

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List the ordered pairs in the equivalence relations produced by these partitions of \(\{0, 1, 2, 3, 4, 5\}\).

a) \(\{0\}, \{1, 2\}, \{3, 4, 5\}\)
\[ \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4)(5, 5)\} \]

b) \(\{0, 1\}, \{2, 3\}, \{4, 5\}\)
\[ \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\} \]

c) \(\{0, 1, 2\}, \{3, 4, 5\}\)
\[ \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\} \]