9.5 Equivalence Relations

A relation on a set A is called an equivalence relation if it is *reflexive*, *symmetric*, and *transitive*.

Equivalence Classes

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class. This can be represented by:

$$[a]_R = \{s|(a, s) \in R\}$$

Partitions and Equivalence Classes

Let $A_1, A_2, \ldots, A_i$ be a collection of subsets of S. Then the collection forms a partition of S if the subsets are nonempty, disjoint and exhaust S:

- $A_i \neq \emptyset$ for $i \in I$
- $A_i \cap A_j = \emptyset$ if $i \neq j$
- $\bigcup_{i \in I} A_i = S$

Theorem 1: Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

- aRb
- $[a] = [b]$
- $[a] \cap [b] \neq \emptyset$

Theorem 2: Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i | i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes.

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Which of these relations on $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a) $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

This is an equivalence relation because it is reflexive, symmetric, and transitive.

b) $\{(0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2)(3, 3)\}$

This is not an equivalence relation because it is neither reflexive nor transitive. Missing $(1, 1)$ for reflexive and missing $(0, 3)$ for the path $(0, 2), (2, 3)$ for transitive.

c) $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}$

This is an equivalence relation because it is reflexive, symmetric, and transitive.

d) $\{(0,0),(1,1),(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

This is not an equivalence relation because it is not transitive. Missing $(1,2)$ for the path $(1,3),(3,2)$.

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Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x,y) such that f$(x) = f(y)$. Show that R is an equivalence relation on A.

This relation is reflexive because it is obvious that $f(x) = f(x)$ for all $x \in A$.
This relation is symmetric because if $f(x) = f(y)$, then it follows that $f(y) = f(x)$.
This relation is transitive because if we have $f(x) = f(y)$ and $f(y) = f(z)$, then it follows that $f(x) = f(z)$.

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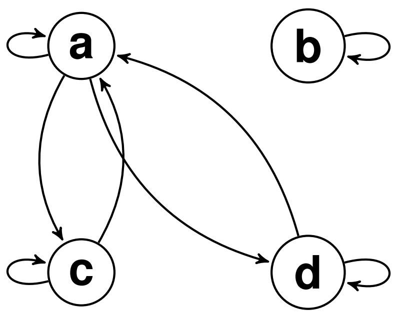
Determine whether the relation with the directed graph shown is an equivalence relation.



Not an equivalence relation because we are missing the edges (c,d) and (d,c) for transitivity.

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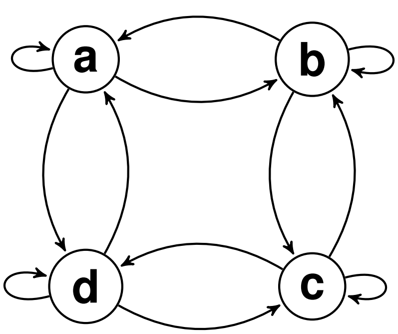
Determine whether the relation with the directed graph shown is an equivalence relation.



Not an equivalence relation because we are missing the edges $(a,c),(c,a),(b,d)$, and (d,b) for transitivity.

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What is the congruence class $[n]_5$ (that is, the equivalence class of n with respect to congruence modulo 5) when n is

a) 2?

$[2]_5 = \{i | i \equiv 2 \pmod 5\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$

b) 3?

$[3]_5 = \{i | i \equiv 3 \pmod 5\} = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$

c) 6?

$[6]_5 = \{i | i \equiv 6 \pmod 5\} = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$

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Which of these collections of subsets are partitions of $\{1, 2, 3, 4, 5, 6\}$?

a) $\{1, 2\}, \{2, 3, 4\}, \{4, 5, 6\}$

Not a partition because these sets are not pairwise disjoint. The element 2 and 4 appear in two of these sets.

b) $\{1\}, \{2, 3, 6\}, \{4\}, \{5\}$

This is a partition.

c) $\{2, 4, 6\}, \{1, 3, 5\}$

This is a partition.

d) $\{1, 4, 5\}, \{2, 6\}$

Not a partition because it is missing the element 3 in any of the sets.

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List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

a) $\{0\}, \{1, 2\}, \{3, 4, 5\}$

$\{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4)(5, 5)\}$

b) $\{0, 1\}, \{2, 3\}, \{4, 5\}$

$\{(0, 0), (0, 1), (1, 0), (1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$

c) $\{0, 1, 2\}, \{3, 4, 5\}$

$\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5),$
$(4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$