2.6 Matrices

A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix.

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$ is a 4×3 matrix.

We specify an element of the matrix with $m_{i,j}$ where the element is at row i and column j. For example, $m_{2,3} = 6$.

Matrix Equality

Matrices are equal if and only if they have the same number of rows, the same number of columns, and the same elements at every index.

Matrix Addition

We can only perform matrix addition if the matrices have the same dimensions. To express the addition of two matrices, A and B, we write $A + B = [a_{i,j} + b_{i,j}]$. Then simply add the values at corresponding indices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} (1+9) & (2+8) & (3+7) \\ (4+6) & (5+5) & (6+4) \\ (7+3) & (8+2) & (9+1) \end{bmatrix} = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}$$

Matrix Multiplication

If A and B are matrices, we can write AB to denote their multiplication. Matrix multiplication is not commutative ($AB \neq BA$).

We can only multiply matrices if and only if the first matrix has the same number of columns as the number of rows in the second matrix.

$AB = C$
A is an $i \times k$ matrix
B is an $k \times j$ matrix
The result C is an $i \times j$ matrix
$C_{i,j} = (a_{i,1}b_{1,j}) + (a_{i,2}b_{2,j}) + (a_{i,2}b_{2,j}) + \ldots + (a_{i,k}b_{k,j})$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} (1 \cdot 5 + 2 \cdot 7) & (1 \cdot 6 + 2 \cdot 8) \\ (3 \cdot 5 + 4 \cdot 7) & (3 \cdot 6 + 4 \cdot 8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Identity Matrix

The identity matrix is a $n \times n$ square matrix where the main diagonal consist of all ones and zeros elsewhere.

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Inverse Matrix

For a square matrix A, the inverse written as A^{-1}. If A is multiplied by A^{-1}, then the result is the identity matrix. $AA^{-1} = A^{-1}A = I_n$. Note that not all square matrices have an inverse.

Power of Matrices

Square matrices can be multiplied by themselves repeatedly because they have the same number of rows and columns. An $n \times n$ matrix A raised to the positive integer k is defined as

$$A^k = AAA \dots A$$

$$A^0 = I$$

Transpose of Matrices

The Transpose of A, denoted by A^t, is obtained by turning all the rows into columns and vice versa.

Zero-One Matrices

Zero-one matrices are matrices that only contain 0 or 1.
Join: $A \vee B = [a_{ij} \vee b_{ij}]$
Meet: $A \wedge B = [a_{ij} \wedge b_{ij}]$
Boolean Product: Denoted by $A \odot B$, where $c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \dots \vee (a_{ik} \wedge b_{kj})$

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Find AB if

a) $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} (2 \cdot 0 + 1 \cdot 1) & (2 \cdot 4 + 1 \cdot 3) \\ (3 \cdot 0 + 2 \cdot 1) & (3 \cdot 4 + 2 \cdot 3) \end{bmatrix} = \begin{bmatrix} 1 & 11 \\ 2 & 18 \end{bmatrix}$$

b) $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} (1 \cdot 3 + (-1) \cdot 1) & (1 \cdot 2 + (-1) \cdot 0) & (1 \cdot (-1) + (-1) \cdot 3) \\ (0 \cdot 3 + 1 \cdot 1) & (0 \cdot 2 + 1 \cdot 0) & (0 \cdot (-1) + 1 \cdot 3) \\ (2 \cdot 3 + 3 \cdot 1) & (2 \cdot 2 + 3 \cdot 0) & (2 \cdot (-1) + 3 \cdot 3) \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 & -4 \\ 1 & 0 & 3 \\ 9 & 4 & 7 \end{bmatrix}$$

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Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Find a formula for A^n, whenever n is a positive integer.

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$A^3 = AA^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$
$$A^4 = AA^3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$
$$\vdots$$
$$A^n = AA^{n-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

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Let A be the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that if $ad - bc \neq 0$, then

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ba}{ad-bc} \\ \frac{cd-cd}{ad-bc} & \frac{-bc+ad}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \dfrac{ad-bc}{ad-bc} & \dfrac{db-bd}{ad-bc} \\ \dfrac{-ca+ac}{ad-bc} & \dfrac{-bc+ad}{ad-bc} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Extra Problem

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 4 & 6 \\ 1 & 1 & 3 & 7 \end{bmatrix}$$

What is A^t?

$$A^t = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$$

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Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ Find

a) $A \vee B$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

b) $A \wedge B$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) $A \odot B$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (1 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 1) \\ (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) & (1 \wedge 1) \vee (1 \wedge 1) \vee (0 \wedge 1) \\ (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) & (0 \wedge 1) \vee (0 \wedge 1) \vee (1 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$