4.3 Primes and Greatest Common Divisors

Primes

An integer p greater than 1 is called *prime* if the only positive factors of p are 1 and p. A positive integer that is greater than 1 and is not prime is called *composite*.

The Fundamental Theory of Arithmetic

Every integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Theorem 2

If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$.

Greatest Common Divisor

Let a and b be integers, not both zero. The largest integer d such that $d \mid a$ and $d \mid b$ is called the *greatest common divisor* of a and b. The greatest common divisor of a and b is denoted by $\gcd(a, b)$.

Finding the Greatest Common Divisor using Prime Factorization

Suppose the prime factorizations of a and b are:

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$$

$$b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$$

where each exponent is a nonnegative integer, and where all primes occurring in either prime factorization are included in both, with zero exponents if necessary. Then:

$$\gcd(a, b) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \cdots p_n^{\min(a_n, b_n)}$$

Relatively Prime

The integers a and b are *relatively prime* if their greatest common divisor is 1.

Least Common Multiple

The *least common multiple* of the positive integers a and b is the smallest positive integer that is divisible by both a and b. The least common multiple of a and b is denoted by $\text{lcm}(a, b)$.

Finding the Least Common Multiple Using Prime Factorizations

Suppose the prime factorizations of a and b are:

$$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$$
$$b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$$

where each exponent is a nonnegative integer, and where all primes occurring in either prime factorization are included in both, with zero exponents if necessary. Then:

$$\text{lcm}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \cdots p_n^{\max(a_n,b_n)}$$

Theorem 5

Let a and b be positive integers. Then $ab = \text{gcd}(a,b) \cdot \text{lcm}(a,b)$.

The Euclidean Algorithm

Let $a = bq + r$ where a, b, q, and r are integers. Then $\text{gcd}(a,b) = \text{gcd}(b,r)$. Also written as $\text{gcd}(a,b) = \text{gcd}((b,(a \mod b))$.

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Find the prime factorization of each of these integers.

a) 88

$\sqrt{88} \approx 9.38$
$88/2 = 44$
$44/2 = 22$
$22/2 = 11$
Therefore $88 = 2^3 \cdot 11$

b) 126

$\sqrt{126} \approx 11.22$
$126/2 = 63$
$63/3 = 21$
$21/3 = 7$
Therefore $63 = 2 \cdot 3^2 \cdot 7$

c) 729

$\sqrt{729} = 27$
$729/3 = 243$
$243/3 = 81$
$81/3 = 27$
$27/3 = 9$
$9/3 = 3$
Therefore $729 = 3^6$

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Determine whether the integers in each of these sets are pairwise relatively prime.

a) $11, 15, 19$

$\gcd(11, 16) = 1, \gcd(11, 19) = 1, \gcd(15, 19) = 1$
These numbers are pairwise relatively prime.

b) $14, 15, 21$

$\gcd(15, 21) = 3$. Since $3 \neq 1$, this set is not pairwise relatively prime.

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What are the greatest common divisors of these pairs of integers?

a) $3^7 \cdot 5^3 \cdot 7^3$, $2^{11} \cdot 3^5 \cdot 5^9$

$2^{\min(0,11)} \cdot 3^{\min(7,5)} \cdot 5^{\min(3,9)} \cdot 7^{\min(3,0)}$
$= 3^5 \cdot 5^3$

b) $11 \cdot 13 \cdot 17$, $2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3$

1

c) 23^{31}, 23^{17}

$23^{\min(31,17)} = 23^{17}$

d) $41 \cdot 43 \cdot 53$, $41 \cdot 43 \cdot 53$

$41 \cdot 43 \cdot 53$

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What is the least common multiple of these pairs of integers?

a) $3^7 \cdot 5^3 \cdot 7^3$, $2^{11} \cdot 3^5 \cdot 5^9$

$2^{\max(0,11)} \cdot 3^{\max(7,5)} \cdot 5^{\max(3,9)} \cdot 7^{\max(3,0)}$
$= 2^{11} \cdot 3^7 \cdot 5^9 \cdot 7^3$

b) $11 \cdot 13 \cdot 17$, $2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3$

$11 \cdot 13 \cdot 17 \cdot 2^9 \cdot 3^7 \cdot 5^5 \cdot 7^3$

c) 23^{31}, 23^{17}

$23^{\max(31,17)} = 23^{31}$

d) $41 \cdot 43 \cdot 53$, $41 \cdot 43 \cdot 53$

$41 \cdot 43 \cdot 53$

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Find gcd(92928, 123552) and lcm(92928, 123552) and verify that $\text{gcd}(92928, 123552) \cdot \text{lcm}(92928, 123552) = 92928 \cdot 123552$. [Hint: First find the prime factorizations of 92928 and 123552.]

$92928 = 2^8 \cdot 3 \cdot 11^2$
$123552 = 2^5 \cdot 3^3 \cdot 11 \cdot 13$

$\text{gcd}(92928, 123552) = 2^5 \cdot 3 \cdot 11$
$\text{lcm}(92928, 123552) = 2^8 \cdot 3^3 \cdot 11^2 \cdot 13$

$\text{gcd}(92928, 123552) \cdot \text{lcm}(92928, 123552) = 92928 \cdot 123552$
$(2^5 \cdot 3 \cdot 11) \cdot (2^8 \cdot 3^3 \cdot 11^2 \cdot 13) = (2^8 \cdot 3 \cdot 11^2) \cdot (2^5 \cdot 3^3 \cdot 11 \cdot 13)$
$2^{13} \cdot 3^4 \cdot 11^3 \cdot 13 = 2^{13} \cdot 3^4 \cdot 11^3 \cdot 13$

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Use the Euclidean algorithm to find

c) $\text{gcd}(1001, 1331)$

$1331 = 1001 \cdot 1 + 330$
$1001 = 330 \cdot 3 + 11$
$330 = 11 \cdot 30 + 0$
$\text{gcd}(1001, 1331) = \text{gcd}(1001, 330) = \text{gcd}(330, 11) = \text{gcd}(11, 0) = 11$

f) $\text{gcd}(9888, 6060)$

$9888 = 6060 \cdot 1 + 3828$
$6060 = 3828 \cdot 1 + 2232$
$3828 = 2232 \cdot 1 + 1596$
$2232 = 1596 \cdot 1 + 636$
$1596 = 636 \cdot 2 + 324$
$636 = 324 \cdot 1 + 312$
$324 = 312 \cdot 1 + 12$
$312 = 12 \cdot 26 + 0$
$\text{gcd}(9888, 6060) = \text{gcd}(6060, 3820) = \text{gcd}(3828, 2232) = gcd(2232, 1596) = \text{gcd}(1596, 636) = \text{gcd}(636, 324) = \text{gcd}(324, 312) = \text{gcd}(32, 12) = \text{gcd}(12, 0) = 12$