1.3 Propositional Equivalences

Tautologies, Contradictions, and Contingencies

- A *tautology* is a compound proposition which is always true.
- A *contradiction* is a compound proposition which is always false.
- A *contingency* is a compound proposition which is neither a tautology nor a contradiction.

Logical Equivalences

Identity	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity Laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivlances Involving Conditional Statements
$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv q \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences Involving Biconditional Statements
$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

We can construct new logical equivalences by applying known logically equivalent statements to show that $A \equiv B$.

Recall that two propositions p and q are logically equivalent if and only if $p \leftrightarrow q$ is a tautology (a.k.a. their truth tables match). However, for very long or complex propositions, it might be less work to do a proof of logical equivalence.

Goal: Get both sides to be the same.
Strategy:

- Apply rules from the list of Logical Equivalences to manipulate one side of the proposition
- Apply one rule per line
- Keep applying rules until we arrive at our goal

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Use De Morgan's laws to find the negation of each of the following statements.

a) Jan is rich and happy.

p = "Jan is rich"
q = "Jan is happy"
$p \wedge q$
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
"Jan is not rich, or not happy."

b) Mei walks or takes the bus to class.

p = "Mei walks to class"
q = Mei takes the bus to class."
$p \vee q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$
"Mei does not walk to class, and Mei does not take the bus to class."

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Show that each conditional statement is a tautology without using truth tables

b $p \to (p \vee q)$

$p \to (p \vee q)$	
$\equiv \neg p \vee (p \vee q)$	Law of Implication
$\equiv (\neg p \vee p) \vee q$	Associative Law
$\equiv \mathbf{T} \vee q$	Negation Law
$\equiv \mathbf{T}$	Domination law

d $(p \wedge q) \rightarrow (p \rightarrow q)$

$(p \wedge q) \rightarrow (p \rightarrow q)$	
$\equiv \neg(p \wedge q) \vee (p \rightarrow q)$	Law of Implication
$\equiv \neg(p \wedge q) \vee (\neg p \vee q)$	Law of Implication
$\equiv (\neg p \vee \neg q) \vee (\neg p \vee q)$	De Morgan's Law
$\equiv (\neg p) \vee (\neg q \vee (\neg p \vee q))$	Associative Law
$\equiv (\neg p) \vee ((\neg p \vee q) \vee \neg q)$	Commutative Law
$\equiv (\neg p) \vee (\neg p \vee (q \vee \neg q))$	Associative Law
$\equiv (\neg p) \vee (\neg p \vee \mathbf{T})$	Negation Law
$\equiv (\neg p) \vee (\mathbf{T})$	Domination Law
$\equiv \mathbf{T}$	Domination Law

f $\neg(p \rightarrow q) \rightarrow \neg q$

$\neg(p \rightarrow q) \rightarrow \neg q$	
$\equiv \neg\neg(p \rightarrow q) \vee \neg q$	Law of Implication
$\equiv (p \rightarrow q) \vee \neg q$	Double Negation
$\equiv (\neg p \vee q) \vee \neg q$	Law of Implication
$\equiv \neg p \vee (q \vee \neg q)$	Associative Law
$\equiv \neg p \vee \mathbf{T}$	Negation Law
$\equiv \mathbf{T}$	Domination Law

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Determine whether $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p)$ is a tautology.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \wedge (p \rightarrow q))$	$(\neg q \wedge (p \wedge q)) \rightarrow \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

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Show that $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent.

p	q	$\neg q$	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$	$p \leftrightarrow \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	F	T	T
F	F	T	T	F	F