5.3 Recursive Definitions

Recursion is the general term for the practice of defining an object in terms of itself or of part of itself.

Recursively Defined Functions

A recursive or inductive definition of a function consists of two steps.

- Basis Step: Specify the value of the function at initial values. (e.g. $f(0)$ defined)
- Recursive Step: Give a rule for finding its value at an integer from its values at smaller integers. (For $n > 0$, define $f(n)$ in terms of $f(0), f(1), \ldots, f(n-1)$)

Strings

The set Σ^* of strings over the alphabet Σ:

- Basis Step: $\lambda \in \Sigma^*$ (λ is the empty string)
- Recursive Step: $((w \in \Sigma^*) \wedge (x \in \Sigma)) \rightarrow wx \in \Sigma^*$

String Concatenation

Two strings can be combined via the operation of concatenation. Let Σ be the set of symbols and Σ^* be the set of strings formed from the symbols in Σ. We can define concatenation of two strings as follows.

- Basis Step: If $w \in \Sigma^*$, then $w\lambda = w$.
- Recursive Step: If $w_1 \in \Sigma^*$ and $w_2 \in \Sigma^*$ and $x \in \Sigma$, then $w_1(w_2x) = (w_1w_2)x$.

Structural Induction

To prove a property of the elements of a recursively defined set, we use *structural induction*.

- Basis Step: Show that the result holds for all elements specified in the basis step of the recursive definition.
- Recursive Step: Show that if the statement is true for each of the elements used to construct new elements in the recursive step of the definition, the result holds for these new elements.

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Find $f(1), f(2), f(3)$, and, $f(4)$ if $f(n)$ is defined recursively by $f(0) = 1$ and for $n = 0, 1, 2, \ldots$

a) $f(n+1) = f(n) + 2$

$f(1) = f(0+1) = f(0) + 2 = 1 + 2 = 3$
$f(2) = f(1+1) = f(1) + 2 = 3 + 2 = 5$
$f(3) = f(2+1) = f(2) + 2 = 5 + 2 = 7$
$f(4) = f(3+1) = f(3) + 2 = 7 + 2 = 9$

b) $f(n+1) = 3f(n)$

$f(1) = f(0+1) = 3f(0) = 3(1) = 3$
$f(2) = f(1+1) = 3f(1) = 3(3) = 9$
$f(3) = f(2+1) = 3f(2) = 3(9) = 27$
$f(4) = f(3+1) = 3f(3) = 3(27) = 81$

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Give a recursive definition of the sequence $\{a_n\}$, $n = 1, 2, 3, \ldots$ if

a) $a_n = 6n$

$a_1 = 6$
$a_2 = 12$
$a_3 = 18$
$a_4 = 24$
$a_{n+1} = a_n + 6$, $a_1 = 6$

b) $a_n = 2n + 1$

$a_1 = 3$
$a_2 = 5$
$a_3 = 7$
$a_4 = 9$
$a_{n+1} = a_n + 2$, $a_1 = 3$

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Give a recursive definition of

a) the set of even integers.

Basis Step: $0 \in S$
Recursive Step: if $x \in S$, then $x + 2 \in S$ and $x - 2 \in S$

b) the set of positive integers congruent to 2 modulo 3.

Basis Step: $2 \in S$
Recursive Step: if $x \in S$, then $x + 3 \in S$

c) the set of positive integers not divisible by 5.

Basis Step: $1, 2, 3, 4 \in S$
Recursive Step: if $x \in S$, then $x + 5 \in S$

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Let S be the subset of the set of ordered pairs of integers defined recursively by

- Basis Step: $(0, 0) \in S$
- Recursive Step: If $(a, b) \in S$, then $(a, b+1) \in S$, $(a+1, b+1) \in S$, and $(a+2, b+1) \in S$.

a) List the elements of S produced by the first four applications of the recursive definition.

First Application of Recursive step to S
Applying recursive step to (0,0) gives (0,1), (1,1), (2,1)
So $S = \{(0, 0), (0, 1), (1, 1), (2, 1)\}$

Second Application of Recursive step to S
Omit the application of the Recursive step to (0,0) again, since we already have those terms in S.
Applying recursive step to (0,1) gives (0,2), (1,2), (2,2)
Applying recursive step to (1,1) gives (1,2), (2,2), (3,2)
Applying recursive step to (2,1) gives (2,2), (3,2), (4,2)
So $S = \{(0, 0), (0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2), (3, 2), (4, 2)\}$

Third Application of Recursive step to S
Omit the application of the Recursive step to (0,0),(0,1),(1,1),(2,1) again, since we already have those terms in S.
Applying recursive step to (0,2) gives (0,3), (1,3), (2,3)
Applying recursive step to (1,2) gives (1,3), (2,3), (3,3)
Applying recursive step to (2,2) gives (2,3), (3,3), (4,3)
Applying recursive step to (3,2) gives (3,3), (4,3), (5,3)
Applying recursive step to (4,2) gives (4,3), (5,3), (6,3)
So S={(0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,2), (3,3), (4,2), (4,3), (5,3), (6,3)}

Fourth Application of Recursive step to S
Omit the application of the Recursive step to (0,0),(0,1),(0,2),(1,1),(1,2),(2,1),(2,2),(3,2),(4,2) again, since we already have those terms in S.
Applying recursive step to (0,3) gives (0,4), (1,4), (2,4)
Applying recursive step to (1,3) gives (1,4), (2,4), (3,4)
Applying recursive step to (2,3) gives (2,4), (3,4), (4,4)
Applying recursive step to (3,3) gives (3,4), (4,4), (5,4)
Applying recursive step to (4,3) gives (4,4), (5,4), (6,4)

Applying recursive step to (5,3) gives (5,4), (6,4), (7,4)
Applying recursive step to (6,3) gives (6,4), (7,4), (8,4)
So S={(0,0), (0,1), (0,2), (0,3), (0,4), (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4), (4,2), (4,3), (4,4), (5,3), (5,4), (6,3), (6,4), (7,4), (8,4)}

c) Use structural induction to show that $a \leq 2b$ whenever $(a, b) \in S$.

Inductive Hypothesis: $P(a, b) = a \leq 2b$ whenever $(a, b) \in S$.
Basis Step: $P(0, 0) = 0 \leq 0$
Recursive Step:
$P(a, b + 1) = a \leq 2(b + 1) = \boldsymbol{a \leq 2b + 2}$
$a \leq 2b$ (our inductive hypothesis)
$\boldsymbol{a} \leq 2b \leq \boldsymbol{2b + 2}$

$P(a + 1, b + 1) = a + 1 \leq 2(b + 1) = \boldsymbol{a + 1 \leq 2b + 2}$
$a \leq 2b$ (our inductive hypothesis)
$a + 2 \leq 2b + 2$ (add 2 to both sides)
$\boldsymbol{a + 1} \leq a + 2 \leq \boldsymbol{2b + 2}$

$P(a + 2, b + 1) = a + 2 \leq 2(b + 1) = \boldsymbol{a + 2 \leq 2b + 2}$
$a \leq 2b$ (our inductive hypothesis)
$\boldsymbol{a + 2 \leq 2b + 2}$ (add 2 to both sides)

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Give a recursive definition of w^i, where w is a string and i is a nonnegative integer. (Here w^i represents the concatenation of i copies of the string w.)

Basis Step: $w^0 = \lambda$
Recursive Step: $w^n = w(w^{n-1})$