5.1 Mathematical Induction

The Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

- **Basis Step:** We verify that $P(1)$ is true.
- **Inductive Step:** We show that the conditional statement $P(k) \to P(k+1)$ is true for all positive integers k.

Outline of an Inductive Proof

Let us say we want to prove $\forall n \geq b, P(n)$ where $b \in \mathbb{Z}$

- Do the base case (or basis step): Prove $P(b)$.
- Do the inductive step: Prove $\forall k \geq b, P(k) \to P(k+1)$.
  - E.g. you could use a direct proof as follows:
  - Let $k \geq b$, assume $P(k)$. (inductive hypothesis)
  - Now, under this assumption, prove $P(k+1)$.
- The inductive inference rule then gives us $\forall n \geq b, P(n)$.

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Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

Proof by Induction on n.
Basis step: $n = 0$
$(2 \cdot 0 + 1)^2 = (0+1)(2 \cdot 0 + 1)(2 \cdot 0 + 3)/3$
$1^2 = 1 \cdot 1 \cdot 3/3$
$1 = 1$

Inductive Step: Assume that $n = k$.
Inductive Hypothesis: $1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 = \dfrac{(k+1)(2k+1)(2k+3)}{3}$
Prove that $1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2k+3)^2 = \dfrac{(k+2)(2k+3)(2k+5)}{3}$

LHS: $1^2 + 3^2 + 5^2 + \cdots + (2k+1)^2 + (2k+3)^2$
$= \dfrac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$ by inductive hypothesis
$= (2k+3)[\dfrac{(k+1)(2k+1)}{3} + (2k+3)]$

$= (2k+3)\dfrac{(k+1)(2k+1)+3(2k+3)}{3}$
$= (2k+3)\dfrac{(2k^2+3k+1)+(6k+9)}{3}$
$= \dfrac{(2k+3)(2k^2+9k+10)}{3}$
$= \dfrac{(2k+3)(2k+5)(k+2)}{3}$
Therefore, $1^2+3^2+5^2+\cdots+(2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ for all $n \geq 0$.

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Prove that $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n = 3(5^{n+1}-1)/4$ whenever n is a nonnegative integer.

Proof by Induction on n.
Basis Step: $n = 0$
$3\cdot 5^0 = 3(5^{0+1}-1)/4$
$3 = 3(5-1)/4$
$3 = 3$

Inductive step: Assume that $n = k$
Inductive Hypothesis: $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k = \dfrac{3(5^{k+1}-1)}{4}$
Prove that $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k+3\cdot 5^{k+1} = \dfrac{3(5^{k+2}-1)}{4}$

LHS: $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^k+3\cdot 5^{k+1}$
$= \dfrac{3(5^{k+1}-1)}{4} + 3\cdot 5^{k+1}$ by inductive hypothesis
$= \dfrac{3\cdot 5^{k+1}-3}{4} + 3\cdot 5^{k+1}$
$= \dfrac{3\cdot 5^{k+1}-3+4\cdot 3\cdot 5^{k+1}}{4}$
$= \dfrac{3(5^{k+1}-1+4\cdot 5^{k+1})}{4}$
$= 3[\dfrac{5^{k+1}+4\cdot 5^{k+1}}{4} - \dfrac{1}{4}]$
$= 3[\dfrac{(5^{k+1})(1+4)}{4} - \dfrac{1}{4}]$
$= 3[\dfrac{(5^{k+1})(5)}{4} - \dfrac{1}{4}]$
$= 3[\dfrac{5^{k+2}}{4} - \dfrac{1}{4}]$
$= \dfrac{3(5^{k+2}-1)}{4}$
Therefore, $3+3\cdot 5+3\cdot 5^2+\cdots+3\cdot 5^n = 3(5^{n+1}-1)/4$ for all $n \geq 0$.