Representation of Integers (lecture)

ICS312

Machine-Level and Systems Programming

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Integer Representation

- A computer needs to store integers in memory/registers
- Stored using different numbers of bytes (1 byte = 8 bits):
  - 1-byte: “byte”
  - 2-byte: “half word” (or “word”)
  - 4-byte: “word” (or “double word”)
  - 8-byte: “double word” (or “paragraph”, or “quadword”)
  - Different computers have used different word sizes, so it’s always a bit confusing to just talk about a “word” without any context
- Regardless of the number of bytes, integers are stored in binary, so we need a method to encode integers in binary
  - It’s more subtle than you think
- But before we get there, let’s talk about mathematics vs. computer arithmetic….

Math vs. Computer Arithmetic

- In math, we add two numbers and the result is whatever it is
- In computer arithmetic, we specify the size of the numbers on which we perform the operation, e.g., a 16-bit addition
- And the result must then be 16-bit: if we have a leftover carry that would make the result 17-bit, the carry is dropped!

Math

```
    F111
+   6111
  ------
   15222
```

Computer Arithmetic

```
    F111
+   6111
  ------
    5222
```

Math vs. Computer Arithmetic

Math

```
  F111
+ 6111
------
 15222
```

Computer Arithmetic

```
  F111
+ 6111
------
 5222
```

- Even weirder: sometimes the above will be considered numerically correct, and sometimes it will be considered numerically incorrect
- Anybody knows how we call it when “dropping the carry” makes the numerical result incorrect?

Unsigned and Signed Numbers

- From now on, based on the previous slides, we'll always specify the number of bytes for a value (and thus for the operation being performed)
  - We'll never say "we add 12 and 4"
  - We'll say "we add 2-byte value 12 to 2-byte value 4"

- So how do we encode integers?

- Integers come in two flavors:
  - **Unsigned**: **ONLY positive** values from 0 to 2^b-1
  - **Signed**: **positive AND negative** values, with about the same number of negative values as the number of positive values

Unsigned / Signed Integers

- In some languages you can declare integers as signed or unsigned depending on what you need
  - If you know a variable will only be positive, then you have a higher maximum value when using unsigned
  - Signedness is important when working at the bit level (see much later in the semester)
  - The compiler/IDE can help a little bit by throwing a warning when you assign a negative value to an unsigned number

```
// C/C++
int x = -12;              // signed
signed int y = 40;        // signed
unsigned int z = 23;      // unsigned
```

```
// Rust
let x: i16 = -12;    // signed 16-bit
let y: u32 = 2_031; // unsigned 32-bit
                    // (note the convenient _ that
                    // acts as a comma)
```

Unsigned / Signed Integers

```
// C/C++
int x = -12;           // signed
signed int y = 40;     // signed
unsigned int z = 40;   // unsigned
```

```
// Rust
let x: i16 = -12; // signed
let y: u32 = 2_031; // unsigned
```

- In Java, Python, JavaScript all integers are signed (there is no unsigned data type), which has raised A LOT of complaints
- But these languages have APIs to perform unsigned arithmetic

```
// Java
Integer.divideUnsigned(-100, -12) // divide as if numbers were unsigned

// Python
ctypes.c_uint32(-10).value        // interpret -10 as unsigned (32-bit)

// JavaScript
x = -10 >>> 0                     // interpret -10 as unsigned
```

- The code above is likely confusing right now because we don't know yet how we encode signed/unsigned integers in binary… stay tuned (we'll come back to these three examples!)

Encoding Unsigned Integers

- **Encoding unsigned integers is easy:** just use the bits of the integer's binary representation
- Example: 1-byte unsigned number 33_{10} is encoded as 00100001_2 (21_{16})

- **That's all!**

- Just note that we show exactly 8 bits, which may include the leading zeros
  - In mathematics, we typically don't show leading zeros
  - But now we're in the world of compute arithmetic
- So, if I say, what's zero in binary as an 8-bit number, I should write 00000000 not just 0

Encoding Signed Integers

- **Encoding signed integer raises a question:** how to store the sign?
- One approach is called sign-magnitude: reserve the leftmost bit to represent the sign

  00100101 denotes + 0100101_2

  10100101 denotes - 0100101_2

- It's very easy to negate a number: just flip the leftmost bit
- Unfortunately, sign-magnitude complicates the logic of the CPU
  - There are two representations for zero: 10000000 and 00000000
  - Some operations are thus more complicated to implement in hardware
  - See a computer architecture / engineering course

One's complement

- Another idea to encode a negative number is to take the complement (i.e., flip all bits) of its positive counterpart
- Example: I want to encode integer -87
  - $87_{10} = 01010111_2$
  - $-87_{10} = 10101000$
- Simple, but still two representations for zero: 00000000 and 11111111
- It turns out that computer logic to deal with 1's complement arithmetic is complicated
- **Important:** it's easy to compute the 1's complement of a number represented in hexadecimal
  - let's consider: 57_{16}
  - subtract each hex digit from F: F-5=A, F-7=8
  - 1's complement of 57_{16} is $A8_{16}$

Two's complement

- While sign-magnitude and 1's complement were used in older computers, nowadays all computers use 2's complement to encode signed integers
- Computing the 2's complement representation of a negative number is done in **two steps** (**"flip and add one"**)
  - Compute the 1's complement of the positive version of the number
  - Add 1 to the result
  - The gives the representation of the negative number

Two's complement: example

- Let's encode -87_{10}
  - First, start with the positive version of the number: $87_{10} = 57_{16}$
  - "Flip" the bits or hex digits to compute the one's complement: $A8_{16}$
  - Add one: $A9_{16}$
- Let's invert again to check we get back to the positive number
  - We start with: $A9_{16}$
  - Flip the digits (one's complement): 56_{16}
  - Add one: 57_{16}, which represents 87_{10}
- So, when I write, say in C++ `char x = -87;` somewhere in RAM the value A9 (or 10101001) is stored

Two things to note

- **Thing #1: There is a single representation for zero!**
  - Assuming 8-bit signed numbers, zero is 0000 0000
  - Let's compute -0:
    - Flip: 1111 1111
    - Add one: 1 0000 0000 (9 bits!!)
  - BUT, when adding two X-bit quantities in a computer one **always** obtains another X-bit quantity
    - Key difference between arithmetic and **computer** arithmetic
  - The computer DROPS the extra carry because it doesn't "fit"
  - Final result: 0000 0000
  - And so, there is a single representation for 0 (unlike for 1's complement)

- **Thing #2: -1's representation is all bits set to 1**
  - +1 is represented as 0000 0001
  - Flip: 1111 1110
  - Add one: 1111 1111

How to tell the sign of a signed integer?

- All programming languages support signed integers (as far as I know)
- A very common need is to determine whether a signed value is positive or negative
  - Whenever I write code like: `if (x > 0) {…}` then the compiler has to generate code that does the test
  - As humans debugging (assembly) code, we'll look at bytes in registers or RAM and will need to tell whether some value is positive or negative

- **The most significant bit (the leftmost bit) indicates the sign of the number (0: positive, 1: negative)**
  - In hex, if the left-most "digit" is 8, 9, A, B, C, D, E, or F, then the number is negative, otherwise it is positive
  - Those are the hex digit of the form 1xxx in binary

- Let's look at ALL 3-bit unsigned and signed numbers….

All 3-bit unsigned/signed numbers

UNSIGNED	
Decimal	Representation
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

SIGNED	
Decimal	Representation
0	000
1	001
2	010
3	011
-4	100
-3	101
-2	110
-1	111

All 3-bit unsigned/signed numbers

UNSIGNED	
Decimal	Representation
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

SIGNED	
Decimal	Representation
0	000
	001
	010
3	011
-4	100
	101
	110
-1	111

Smallest number: 000
Largest number: 111

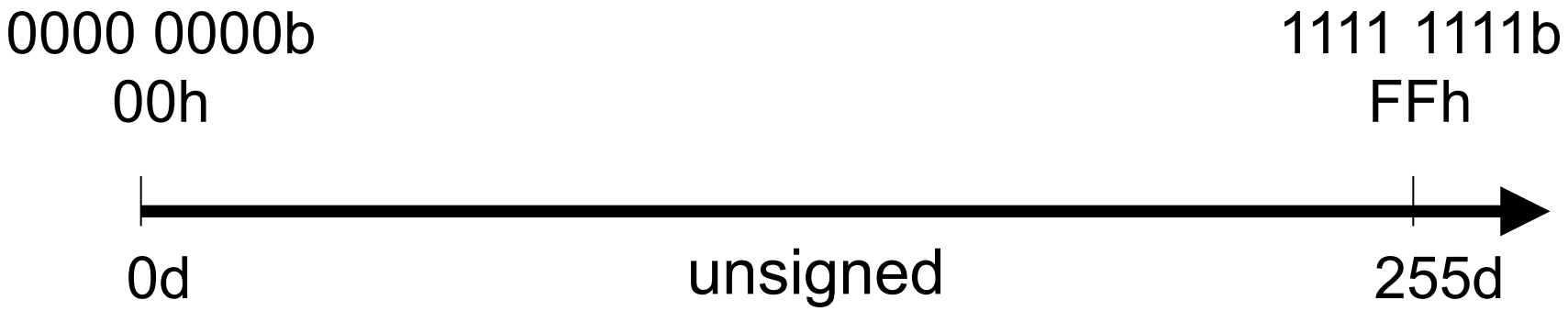
Smallest >= 0 number: 000
Largest >= 0 number: 011

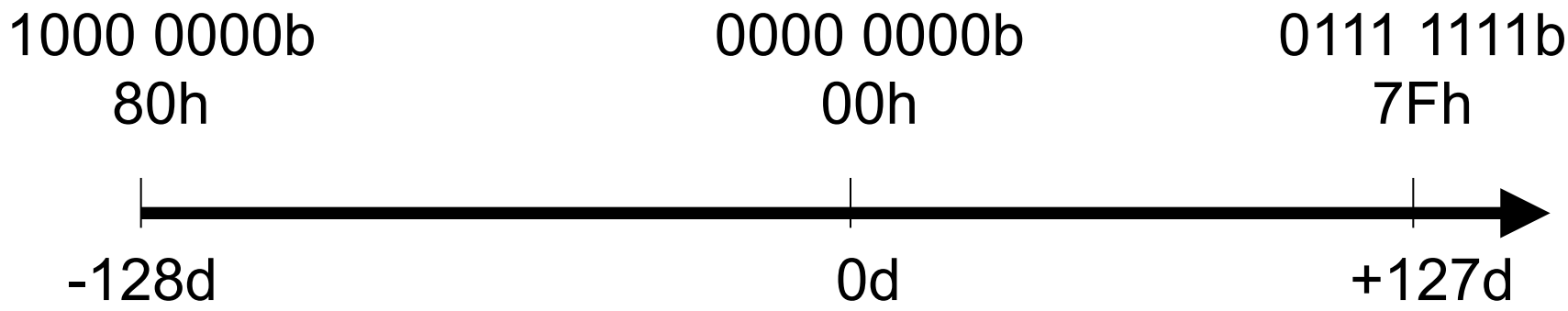
Smallest < 0 number: 100
Largest < 0 number: 111

Ranges of Numbers

- For 1-byte values
  - Unsigned
    - Smallest value: 00_{16} or $0000\ 0000_2$ (0_{10})
    - Largest value: FF_{16} or $1111\ 1111_2$ (255_{10})
  - Signed
    - Smallest value: 80_{16} or $1000\ 0000_2$ (-128_{10})
    - Largest value: $7F_{16}$ or $0111\ 1111_2$ ($+127_{10}$)
- For 2-byte values
  - Unsigned
    - Smallest value: 0000_{16} (0_{10})
    - Largest value: $FFFF_{16}$ ($65{,}535_{10}$)
  - Signed
    - Smallest value: 8000_{16} ($-32{,}768_{10}$)
    - Largest value: $7FFF_{16}$ ($+32{,}767_{10}$)
- etc.

1-byte Ranges





(makes sense it's **even**)

signed

(makes sense it's **odd**)

The magic of 2's complement (for addition)

- Say I have two 1-byte values, A3 and 17, and I add them together:

  $A3_{16} + 17_{16} = BA_{16}$ ("blind" hex addition)

- If my interpretation of the numbers is unsigned:
  - $A3_{16} = 163_{10}$
  - $17_{16} = 23_{10}$
  - $BA_{16} = 186_{10}$
  - and indeed, $163_{10} + 23_{10} = 186_{10}$
- If my interpretation of the numbers is signed:
  - $A3_{16} = -93_{10}$
  - $17_{16} = 23_{10}$
  - $BA_{16} = -70_{10}$
  - and indeed, $-93_{10} + 23_{10} = -70_{10}$
- So, as long as I stick to my interpretation, the binary addition does the right thing assuming 2's complement representation!!!
  - Same thing for the subtraction

Dropping the Carry?

- Remember earlier when I said that dropping the carry can be numerically correct?
  - That should have felt wrong, because we are dropping information
- We'll come back to this but just consider 1-byte signed hex addition: FF + FF
- In math, we'd get: 1FE
- In computer arithmetic we get: FE (carry is dropped)
- So in computer arithmetic: FF + FF = FE
- That makes sense: FF is -1d, and FE is -2d
  - And yes, -1 + -1 = -2 :)
- So dropping the carry is numerically correct!!!
- Stay tuned for more on this later….

The Task of the (Assembly) Programmer

- The computer simply stores data as bits based on what a program does
- **It has no idea what the data means and doesn't know whether numbers are signed or unsigned**
- We, as programmers, have precise interpretations of what bits mean
  - "I store a 4-byte signed integer", "I store a 1-byte integer which is an ASCII code"
- When using a high-level language we can say what data means
  - "I declare x as an `int` and y as an `unsigned char`"

- **When writing assembly code, we don't have any data types**
- **But** we have many instructions that operate on all types of data
- **It's our responsibility to use the instructions that correspond to the data**
- We just saw that addition is the same for both signed and unsigned numbers
  - And therefore there is a single "addition instruction": easy
- But it's not the case for all operations
  - We'll see "signed multiplication" and "unsigned multiplication" instructions

Signed does not mean negative!

- It means “a number that is encoded in binary using 2’s complement so that it can take either positive or negative values”
- The encoding of a positive value is the “normal” one (just the binary representation but only if it starts with a 0 bit - otherwise it’s out of range)
- The encoding of a strictly negative value is the “flip-and-add-one” transformation of the strictly positive counterpart value (and the representation will thus NECESSARILY starts with a 1 bit)

Back to the PL examples

```
// C/C++
int x = -12;            // signed
signed int y = 40;      // signed
unsigned int z = 40;    // unsigned
```

- x is encoded as FF FF FF F4 in hex
- y is encoded as 00 00 00 28 in hex
- z is encoded as 00 00 00 28 in hex

```
// Rust
let x: i16 = -12; // signed
let y: u32 = 2_031; // unsigned
```

- x is encoded as FF F4 in hex
- y is encoded as 00 00 04 07 in hex

Back to the PL examples

```
// Java
Integer.divideUnsigned(-100, -12) // divide as if numbers were unsigned
```

- -100 is a signed number encoded as FF FF FF 9B in hex
- -12 is a signed numbers encoded as FF FF FF F4 in hex
- The above will:
  - Interpret FF FF FF 9B as an unsigned number (which is $4{,}294{,}967{,}196_{10}$)
  - Interpret FF FF FF F4 as an unsigned number (which is $4{,}294{,}967{,}284_{10}$)
  - Perform the division of $4{,}294{,}967{,}196_{10}$ by $4{,}294{,}967{,}284_{10}$
  - The quotient will be zero!

Back to the PL examples

```
// Python
ctypes.c_uint32(-10).value          // interpret -10 as unsigned (32-bit)
```

- -10 is a signed number encoded as FF FF FF F6 in hex
- The above will:
  - Interpret FF FF FF F6 as an unsigned number (which is $4{,}294{,}967{,}286_{10}$)
  - Return the (positive) integer 4,294,967,286

Back to the PL examples

```
// JavaScript
x = -10 >>> 0                          // interpret -10 as unsigned
```

- -10 is a signed number encoded as FF FF FF F6 in hex
- The above will:
  - Interpret FF FF FF F6 as an unsigned number (which is $4{,}294{,}967{,}286_{10}$)
  - Perform a logical right shift by 0 bits, which has the side effect of setting x to $4{,}294{,}967{,}286_{10}$
  - This is very odd (but commonly done in JavaScript to use unsigned numbers)
  - We'll understand it fully when we cover bitwise operations and shifts

Conclusion

- We'll come back to numbers and arithmetic when we use arithmetic assembly instructions
- But for now you must make sure you have solid mastery of the material in this module

- We'll do some of the posted practice problems in-class

- We will then have **a quiz on this module**